ASYMPTOTIC BEHAVIOUR OF ANCESTRAL LINEAGES IN SUBCRITICAL CONTINUOUS-STATE BRANCHING POPULATIONS

Clément Foucart [1] and Martin Möhle [2]

Abstract. Consider the population model with infinite size associated to subcritical continuous-state branching processes (CSBP). We study the ancestral lineages as time goes to the past and show that the flow of ancestral lineages, properly renormalized, converges almost surely to the inverse of a drift-free subordinator whose Laplace exponent is explicit in terms of the branching mechanism. The inverse subordinator is partitioning the current population into ancestral families with distinct common ancestors. When Grey's condition is satisfied, the population comes from a discrete set of ancestors and the ancestral families have i.i.d. sizes distributed according to the quasi-stationary distribution of the CSBP conditioned on non-extinction. When Grey's condition is not satisfied, the population comes from a continuum of ancestors which is described as the set of increase points $\mathscr{S}$ of the limiting inverse subordinator. The proof is based on a general result for stochastically monotone processes of independent interest, which relates θ-invariant measures and θ-invariant functions for a process and its Siegmund dual.

1. Introduction

Continuous-state branching processes (CSBPs) are positive Markov processes satisfying the branching property. They arise as scaling limits of Galton-Watson processes and form a fundamental class of random population models. Their longterm behaviour has received a great deal of attention since the seventies. We refer for instance to the early works of Bingham [Bin76] and Grey [Gre74]. In the seminal work [BLG00], Bertoin and Le Gall showed how to encode a complete genealogy of a random branching population by considering a flow of subordinators $(X_{s,t}(x), s \leq t, x \geq 0)$. In this setting, for any $x \in (0,\infty)$, the process $(X_t(x), t \geq 0) := (X_{0,t}(x), t \geq 0)$ is a CSBP started from x, viewed as the size of the progeny of all individuals lying in the interval $(0, x)$ at time 0.

The initial value x being arbitrarily large, the population in the flow of subordinators representation has an infinite size at all times and all individuals have arbitrarily old ancestors. More precisely, individuals at time s with descendants at time $t > s$ are the locations of the jumps of the subordinator $x \mapsto X_{s,t}(x)$ and the descendants at time t of the individuals in the population at time s are represented by the jump intervals. We shall provide more background on the flow of subordinators in the sequel.

Date: November 09, 2021.

2010 *Mathematics Subject Classification.* 60J80, 60J70, 92D25.

Key words and phrases. branching processes, continuous-state space, inverse subordinators, ancestral lineage, almost sure asymptotics, Siegmund dual, invariant function, invariant measure, stochastic flows.

[1]Université Sorbonne Paris Nord and Paris 8, Laboratoire Analyse, Géométrie & Applications, UMR 7539. Institut Galilée, 99 avenue J.B. Clément, 93430 Villetaneuse, France; foucart@math.univ-paris13.fr

[2]Eberhard Karls Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany; martin.moehle@uni-tuebingen.de

Most works on CSBPs focus on their long-term behaviour forward in time. We refer to Bertoin et al. [BFM08], Duquesne and Labbé [DL14], Labbé [Lab14] and Foucart and Ma [FM19] for studies in the framework of the flow of subordinators. In this article, we are interested in the backward genealogy of the continuous population and how it behaves on the long-term. To the best of our knowledge, fewer works on CSBPs have been done in this direction. We refer however to the works of Lambert [Lam03], Lambert and Popovic [LP13], Johnston and Lambert [JL21+] and Foucart et al. [FMM19]. The representation of the population model through $(X_{s,t}(x), s \leq t, x \geq 0)$ allows one to follow the ancestral lineages *backward in time*. The work [FMM19] initiates the study of the inverse flow $(\hat{X}_{s,t}(x), s \leq t, x \geq 0)$ defined for $s \leq t$ and $x \in (0, \infty)$, as

$$\hat{X}_{s,t}(x) := \inf\{y \geq 0 : \; X_{-t,-s}(y) > x\}. \tag{1.1}$$

This random variable represents the ancestor at time $-t$ of the individual x in the population at time $-s$. From now on, we consider an arrow of time pointing to the past, and call $\hat{X}_t(x) := \hat{X}_{0,t}(x)$, the ancestor at time $t \geq 0$ (backwards) of the individual x of the population at time 0. The two-parameter flow $(\hat{X}_t(x), t \geq 0, x \geq 0)$ is therefore representing the ancestral lineages of the individuals in the current population. We call $(\hat{X}_t(x), t \geq 0)$ the *ancestral lineage process*. This is a Feller process with no positive jumps. Moreover, for any $x \neq y$, whenever $(\hat{X}_t(x), t \geq 0)$ and $(\hat{X}_t(y), t \geq 0)$ meet, they coalesce and such a coalescence represents the occurrence in the past of a common ancestor of the individuals x and y. We refer to [FMM19] for a study of the coalescent processes embedded in the flow $(\hat{X}_t(x), t \geq 0, x \geq 0)$.

For any fixed $x \in (0, \infty)$, the Markov processes $(\hat{X}_t(x), t \geq 0)$ and $(X_t(x), t \geq 0)$ are linked through a duality relationship, called *Siegmund duality*, of the following form: for any $t \geq 0$ and $x, y \in (0, \infty)$

$$\mathbb{P}\big(\hat{X}_t(x) < y\big) = \mathbb{P}\big(x < X_t(y)\big). \tag{1.2}$$

We refer to Siegmund [Sie76] and Clifford and Sudbury [CS85] for a general study of the duality (1.2). See [FMM19, Equation (3.5), Section 3] for the case of CSBPs.

We wish to mention that Siegmund dual processes of discrete branching Markov processes have been already studied by Asmussen and Sigman [AS96, Example 9], Li et al. [LPLG08] and more recently by Pakes [Pak17]. However no mention was made in these works about the genealogical interpretation of the Siegmund duals and their almost sure renormalisation. Besides the fact that our study lies in the continuous state space setting, one of the main contribution of this article will be to provide a genealogical interpretation of the Siegmund duals and their limits.

We focus in this article on subcritical CSBPs. In such a setting, it has been shown in [FMM19] that for all $x \in (0, \infty)$, the Markov process $(\hat{X}_t(x), t \geq 0)$ is transient. The main aim of this article is to obtain an almost sure renormalisation of the inverse flow $\hat{X}$ of all subcritical CSBPs, including those with no quasi-stationary distributions, namely those for which Grey's condition does not hold, see Section 2. The limit process is the inverse of a certain subordinator with an explicit Laplace exponent. Last but not least, we establish that this inverse subordinator is partitioning the current population into ancestral families with distinct common ancestors.

The paper is organised as follows. Further background on CSBPs and their representation in terms of flow of subordinators are provided in Section 2. Fundamental properties of the ancestral lineage process $(\hat{X}_t(x), t \geq 0)$, such as its Siegmund duality relation with the CSBP $(X_t(x), t \geq 0)$ and the representation of its semigroup, are also recalled. Our main results are stated in Section 3 and proven in Section 4. The proof is based on a general result, established in Theorem 4.1, for stochastically monotone Markov processes by showing how to link (infinite) θ-invariant measures of a process $(X_t, t \geq 0)$ with (increasing) θ-invariant functions of its Siegmund dual process $(\hat{X}_t, t \geq 0)$. We apply this result in the setting of CSBPs.

2. Background on CSBPs and the flow of subordinators

We first recall basic definitions and properties of CSBPs and their representation in terms of flows. These processes are continuous time and continuous space analogue of Galton-Watson Markov chains. They have been introduced by Lamperti [Lam67] and Jiřina [Jiř58]. CSBPs are positive Markov processes satisfying the branching property: for any $x, y \geq 0$ and fixed time $t \geq 0$,

$$X_t(x+y) = X'_t(x) + X''_t(y), \tag{2.3}$$

where $(X_t(x+y), t \geq 0)$ is a CSBP started from $x+y$, and $(X'_t(x), t \geq 0)$ and $(X''_t(y), t \geq 0)$ are two independent copies of the process started respectively from x and y. We refer the reader to [Li11, Chapter 3] for an introduction to CSBPs. Denote by $\mathcal{L}$ the generator of $(X_t(x), t \geq 0)$. For any $q \geq 0$, set $e_q(x) := e^{-qx}$ for any $x \geq 0$. The operator $\mathcal{L}$ acts on the exponential functions as follows. For all $q, x \geq 0$,

$$\mathcal{L} e_q(x) = \Psi(q) x e_q(x), \tag{2.4}$$

where Ψ is a Lévy-Khintchine function and is called the branching mechanism. We refer e.g. to Silverstein [Sil68]. The linear span of exponential functions $A := \mathrm{Span}(\{e_q, q \in [0,\infty)\})$ is a core for generator $\mathcal{L}$.

We shall merely be interested in subcritical CSBPs for which Ψ is of the form

$$\Psi(u) = \frac{\sigma^2}{2}u^2 + \gamma u + \int_0^\infty (e^{-ux} - 1 + ux)\pi(\mathrm{d}x) \text{ for all } u \geq 0, \tag{2.5}$$

where $\gamma = \Psi'(0+) > 0$, $\sigma \geq 0$, and π is a Lévy measure, i.e. a Borel measure such that $\int_0^\infty (x \wedge x^2)\pi(\mathrm{d}x) < \infty$. We assume that either $\pi \neq 0$ or $\sigma > 0$, so that Ψ is not linear. The semigroup of $(X_t(x), t \geq 0)$ satisfies for any $\lambda \in (0, \infty)$, $t \geq 0$ and $x \in [0, \infty)$

$$\mathbb{E}[e^{-\lambda X_t(x)}] = e^{-x v_t(\lambda)}, \tag{2.6}$$

with for any $\lambda \in (0,\infty)$, $t \mapsto v_t(\lambda)$ defined as the solution to the integral equation

$$\int_{v_t(\lambda)}^{\lambda} \frac{\mathrm{d}u}{\Psi(u)} = t. \tag{2.7}$$

Note that $t \mapsto v_t(\lambda)$ solves $\frac{\mathrm{d}}{\mathrm{d}t} v_t(\lambda) = -\Psi(v_t(\lambda))$ with $v_0(\lambda) = \lambda$. As a first consequence of (2.6), the process $(X_t(x), t \geq 0)$ is extinct at time t with probability $e^{-x v_t(\infty)}$ where $v_t(\infty) := \lim\limits_{\lambda \to \infty} v_t(\lambda) \in (0, \infty]$. The latter is finite if and only if Ψ satisfies Grey's condition

$$\int^\infty \frac{\mathrm{d}u}{\Psi(u)} < \infty. \tag{2.8}$$

Lambert [Lam07] and Li [Li00] have studied the subcritical CSBP $(X_t(x), t \geq 0)$ conditioned on non-extinction and established the following weak convergence when (2.8) holds:

$$\mathbb{P}(X_t(x) \in \cdot \,|X_t(x) > 0) \underset{t\to\infty}{\longrightarrow} \nu_\infty(\cdot),$$

where ν_∞, the so-called quasi-stationary distribution of the CSBP, has Laplace transform

$$\int_0^\infty e^{-qx}\nu_\infty(\mathrm{d}x) = 1 - \kappa_\infty(q) := 1 - e^{-\Psi'(0+)\int_q^\infty \frac{\mathrm{d}u}{\Psi(u)}},\ q \geq 0. \tag{2.9}$$

An interesting phenomenon for CSBPs is that when Grey's condition (2.8) does not hold, the latters are persistent in the sense that although subcritical, they are not getting absorbed at 0, but are decreasing towards 0 while keeping positive mass at all times. In particular, no quasi-stationary distributions exist in this setting.

The branching property (2.3) can be translated in terms of independence and stationarity of the increments of the process $(X_t(x), x \geq 0)$ for any fixed time t. The latter is therefore a subordinator and according to (2.6), its Laplace exponent is $\lambda \mapsto v_t(\lambda)$. Starting from this observation, Bertoin and Le Gall in [BLG00] showed that a complete population model can be associated to CSBPs through a flow of subordinators $(X_{s,t}(x), s \leq t, x \geq 0)$.

More precisely, the collection of processes $(X_{s,t}(x), s \leq t, x \geq 0)$ is satisfying the following properties:

(1) For every $s \leq t$, $x \mapsto X_{s,t}(x)$ is a càdlàg subordinator with Laplace exponent $\lambda \mapsto v_{t-s}(\lambda)$.
(2) For every $t \in \mathbb{R}$, $(X_{r,s}, r \leq s \leq t)$ and $(X_{r,s}, t \leq r \leq s)$ are independent.
(3) For every $r \leq s \leq t$, $X_{r,t} = X_{s,t} \circ X_{r,s}$ a.s..

The two-parameter flow $(X_t(x), x \geq 0, t \geq 0) := (X_{0,t}(x), x \geq 0, t \geq 0)$ is a flow of CSBPs with branching mechanism Ψ, each starting from an initial population of arbitrarily large size x. The three-parameter flow above provides a complete genealogy of the underlying (infinite) population: let $y \in (0, \infty)$, if $X_{s,t}(y-) < X_{s,t}(y)$, then the individual y at time s has descendants at time t and those are represented by the interval $\big(X_{s,t}(y-), X_{s,t}(y)\big)$; see Figure 1.

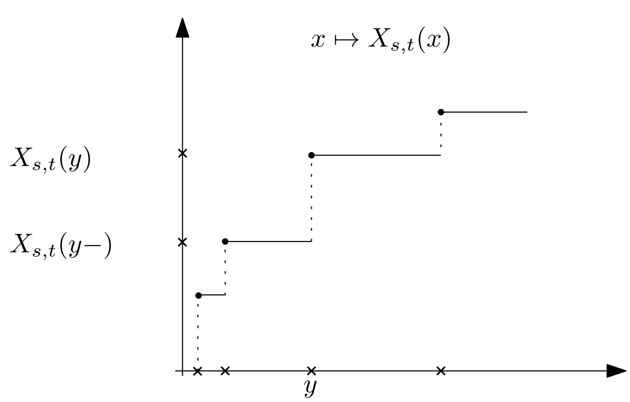

FIGURE 1. Schematic representation of the genealogy forward in time

We will now always work on the probability space on which the flow of subordinators $(X_{s,t}(x), s \leq t, x \geq 0)$ and thus the inverse flow $(\hat{X}_{s,t}(x), t \geq s, x \geq 0)$, see (1.1), are defined. We stress that the statements below hold true with general branching mechanisms, including those that are not subcritical. We shall explicitely mentioned the condition $\Psi'(0+) > 0$ when this is needed. Similarly as for the forward flow, we summarize here fundamental properties of the inverse flow, see [FMM19, Section 3]

(1) For every $t \geq s$, $x \mapsto \hat{X}_{s,t}(x)$ is a càdlàg inverse subordinator.
(2) For every $t \in \mathbb{R}$, $(\hat{X}_{r,s}, r \leq s \leq t)$ and $(\hat{X}_{r,s}, t \leq r \leq s)$ are independent.
(3) For all $r \leq s \leq t$, $\hat{X}_{r,t} = \hat{X}_{s,t} \circ \hat{X}_{s,r}$ a.s..

The ancestral lineage process, defined as

$$(\hat{X}_t(x), t \geq 0) := (\hat{X}_{0,t}(x), t \geq 0),$$

is a non-explosive càdlàg Feller process with no positive jumps, namely for all $x \in (0, \infty)$, $\hat{T}_\infty := \inf\{t > 0 : \hat{X}_t(x) = \infty\} = \infty$ a.s. and $\mathbb{P}\big(\sup_{t>0}(\hat{X}_t(x) - \hat{X}_{t-}(x)) > 0\big) = 0$. Property (3) entails that the flow of processes $\big(\hat{X}_t(x), t \geq 0, x \in (0, \infty)\big)$ is *coalescing*, in the sense that, when two ancestral lineages $(\hat{X}_t(x), t \geq 0)$ and $(\hat{X}_t(y), t \geq 0)$ meet, they merge. Such a coalescence represents the occurrence in the past of a common ancestor of the individuals x and y. We refer the reader to [FMM19, Section 5.2] for a study of coalescent processes embedded in the flow $(\hat{X}_{s,t}, s \leq t)$. The next theorem characterizes the semigroup of $(\hat{X}_t, t \geq 0)$. We denote it by $(Q_t, t \geq 0)$.

Theorem 2.1 (Theorem 3.5, Proposition 3.6 in [FMM19])**.** *For any continuous function f defined on $(0, \infty)$ and any $q > 0$,*

$$\mathbb{E}[Q_t f(\mathbb{e}_q)] = \mathbb{E}[f(\mathbb{e}_{v_t(q)})], \tag{2.10}$$

where for any $\lambda \in (0, \infty)$, $\mathbb{e}_\lambda$ is an exponential random variable with parameter λ. The process $\hat{X}$ admits an entrance boundary at $0+$ if and only if (2.8) is satisfied.

When (2.8) holds, the process started from 0, $(\hat{X}_t(0+), t \geq 0)$, is defined at any time as $\hat{X}_t(0+) := \lim\limits_{x \downarrow 0} \hat{X}_t(x)$ a.s.. This corresponds to the first individual at generation t backwards in time, with descendants at time 0.

We now establish that $\hat{X}$ satisfies some properties of regularity. For $x, y \in (0, \infty)$, set

$$\hat{T}_y := \inf\{t > 0 : \hat{X}_t(x) > y\} = \inf\{t > 0 : \hat{X}_t(x) = y\}. \tag{2.11}$$

We shall sometimes write $\hat{T}_y = \hat{T}_y^x$ to emphasize on the initial state x of the process.

Lemma 2.2 (Regularity)**.** *If $-\Psi$ is not the Laplace exponent of a subordinator, i.e. $\Psi(u) \geq 0$ for some $u > 0$, then the process $\hat{X}$ is regular on $(0, \infty)$, namely for any $x < y$, $\mathbb{P}_x(\hat{T}_y < \infty) > 0$.*

Proof. Let $x, y \in (0, \infty)$, for any $t > 0$, $\mathbb{P}_x(\hat{T}_y < t) \geq \mathbb{P}\big(\hat{X}_t(x) > y\big) = \mathbb{P}\big(x > X_t(y)\big)$. By assumption, $-\Psi$ is not the Laplace exponent of a subordinator, this ensures that the CSBP is not almost surely non-decreasing, see e.g. [Bin76, page 220], and that the event $\{X_t(y) \underset{t\to\infty}{\longrightarrow} 0\}$ has positive probability. Hence, $\mathbb{P}_x(\hat{T}_y < \infty) \geq \lim\limits_{t\to\infty} \mathbb{P}\big(x > X_t(y)\big) \geq \mathbb{P}(X_t(y) \underset{t\to\infty}{\longrightarrow} 0) > 0$. ☐

In the subcritical case, for which $\Psi'(0+)=\gamma>0$, one has $\lim_{t\to\infty} v_t(\lambda)=0$ and all families forward in time are getting extinct. As mentioned in the introduction, in this case the ancestral lineages are transient.

Proposition 2.3 (Proposition 3.8 in [FMM19])**.** *Assume $\Psi'(0+)>0$. For any $x\in(0,\infty)$, the ancestral lineage process $(\hat{X}_t(x), t\geq 0)$ is transient, i.e. $\hat{X}_t(x)\underset{t\to\infty}{\longrightarrow}\infty$ a.s.*

The following versions of Siegmund duality relationships will be more convenient to work with in the sequel. They will allow us to apply a general result, Theorem 4.1, about invariant functions and invariant measures for a given process and its Siegmund dual.

Lemma 2.4. *For any $t\geq 0$ and $x,y\in(0,\infty)$*

$$\{\hat{X}_t(x)>y\}=\{x>X_{-t,0}(y)\} \text{ almost surely,} \tag{2.12}$$

and

$$\mathbb{P}\big(\hat{X}_t(x)>y\big)=\mathbb{P}\big(x>X_t(y)\big). \tag{2.13}$$

Remark 2.5. We stress here on the strict inequalities in (2.13). This form of duality is actually the one used by Siegmund in his fundamental article [Sie76].

Proof. Let $t\geq 0$ and $x,y\in(0,\infty)$. We establish that $\{\hat{X}_t(x)\leq y\}=\{x\leq X_t(y)\}$ almost surely. Note that this is equivalent to (2.12). According to [FMM19, Lemma 3.3-(1), Section 3], $\{\hat{X}_t(x)<y\}=\{x<X_t(y)\}$ almost surely, therefore we only need to focus on the events $\{\hat{X}_t(x)=y\}$ and $\{X_t(y)=x\}$. Recall the definition of $\hat{X}_t(x)$ in (1.1),

$$\{\hat{X}_t(x)=y\}=\{X_{-t,0}(y-)\leq x<X_{-t,0}(y)\}\cup\{X_{-t,0}(y-)=X_{-t,0}(y)=x\}.$$

Since $X_{-t,0}$ is a subordinator, it has no almost sure fixed discontinuities and the event $\{X_{-t,0}(y-)<X_{-t,0}(y)\}$ for fixed y has probability 0. Thus

$$\{\hat{X}_t(x)=y\}=\{X_{-t,0}(y)=x\} \text{ a.s,}$$

and (2.12) is established. Since $X_{-t,0}(y)$ has the same law as $X_t(y)$,

$$\mathbb{P}(\hat{X}_t(x)=y)=\mathbb{P}(X_{-t,0}(y)=x)=\mathbb{P}(X_t(y)=x),$$

and the identity (2.13) holds. □

3. Results

Assume that the branching mechanism Ψ is subcritical, namely $\Psi'(0+)>0$. For any $\lambda\in(0,\infty)$, define the map on $[0,\infty)$:

$$\kappa_\lambda:\theta\mapsto e^{-\Psi'(0+)\int_\theta^\lambda \frac{\mathrm{d}u}{\Psi(u)}}. \tag{3.14}$$

We shall see that κ_λ is the Laplace exponent of a drift-free subordinator, namely it takes the form $\kappa_\lambda(\theta)=\int_0^\infty(1-e^{-\theta x})\nu_\lambda(\mathrm{d}x)$, for any $\theta\geq 0$, where ν_λ is a Lévy measure, i.e. a Borel measure on $(0,\infty)$ such that $\int_0^\infty(1\wedge x)\nu_\lambda(\mathrm{d}x)<\infty$. The latter is finite when $\kappa_\lambda(\infty)<\infty$. This occurs if and only if Grey's condition holds, $\int^\infty\frac{\mathrm{d}u}{\Psi(u)}<\infty$, see (2.8). In this case, the function κ_∞ defined in (2.9), is the Laplace exponent of a compound Poisson process with jump law ν_∞, the quasi-stationary distribution of the Ψ-CSBP conditioned on non-extinction.

Theorem 3.1. *Assume $\Psi'(0+)>0$. Fix $\lambda\in(0,\infty)$. Then, almost surely*

$$v_t(\lambda)\hat{X}_t(x)\underset{t\to\infty}{\longrightarrow}\hat{W}^\lambda(x),\ \text{for all } x\notin J^\lambda:=\{x>0:\hat{W}^\lambda(x)>\hat{W}^\lambda(x-)\},$$

where the process $\hat{W}^\lambda$ has càdlàg paths and its right-inverse process W^λ, defined for any $y\geq 0$ by

$$W^\lambda(y):=\inf\{x\geq 0:\hat{W}^\lambda(x)>y\},$$

is a drift-free subordinator with Laplace exponent κ_λ defined in (3.14). In addition, the following dichotomy holds:

i) *If $\int^\infty\frac{\mathrm{d}u}{\Psi(u)}<\infty$, then for any $\lambda\in(0,\infty]$ the process $(\hat{W}^\lambda(x),x\geq 0)$ has piecewise constant sample paths almost surely.*
ii) *If $\int^\infty\frac{\mathrm{d}u}{\Psi(u)}=\infty$, then for any $\lambda\in(0,\infty)$ the process $(\hat{W}^\lambda(x),x\geq 0)$ has continuous sample paths almost surely.*

Remark 3.2. In case i), since the Lévy measure ν_λ is finite, $(\hat{W}^\lambda(x),x\geq 0)$ is the inverse of a compound Poisson process with Laplace exponent κ_λ. It is natural to choose $\lambda=\infty$ in which case the following almost-sure convergence holds

$$v_t(\infty)\hat{X}_t(x)\underset{t\to\infty}{\longrightarrow}\hat{W}^\infty(x)\text{ for all } x\notin J^\infty,$$

where $(\hat{W}^\infty(x),x\geq 0)$ is the inverse of a compound Poisson process with Laplace exponent κ_∞, whose Lévy measure is the quasi-stationary law of the CSBP. In case ii), one cannot take $\lambda=\infty$, since when Grey's condition does not hold, $v_t(\infty)=\infty$ for all $t\geq 0$.

Recall π the Lévy measure in the Lévy-Khintchine form (2.5) of Ψ. The following corollary shows that the ancestral lineage process $(\hat{X}_t(x),t\geq 0)$ has an exponential growth when the measure π satisfies an $L\log L$ condition.

Corollary 3.3. *For any $\lambda>0$, $v_t(\lambda)\underset{t\to\infty}{\sim}c_\lambda e^{-\Psi'(0+)t}$ for some constant $c_\lambda>0$ if and only if $\int_1^\infty u\log u\,\pi(\mathrm{d}u)<\infty$. Moreover, under this latter condition, almost surely*

$$e^{-\Psi'(0+)t}\hat{X}_t(x)\underset{t\to\infty}{\longrightarrow}\hat{W}(x),\ \text{for all } x\notin J:=\{x>0:\hat{W}(x)>\hat{W}(x-)\},$$

where $\hat{W}$ is the inverse of a subordinator W with Laplace exponent

$$\kappa:\theta\in[0,\infty)\mapsto\theta e^{-\Psi'(0+)\int_0^\theta\left(\frac{1}{\Psi'(0+)u}-\frac{1}{\Psi(u)}\right)\mathrm{d}u}.$$

Remark 3.4. Corollary 3.3 is reminiscent to results for supercritical CSBPs with an $L\log L$ moment, which have an exponential growth when they are not getting extinct, see [Gre74] and e.g. [Li11, Chapter 3].

Example 3.5. *Let $\gamma>0$. Consider the subcritical Neveu CSBP whose branching mechanism is defined by $\Psi(u):=\gamma(u+1)\log(u+1)$ for all $u\geq 0$. Note that $\Psi'(0+)=\gamma>0$ and $\int^\infty\frac{\mathrm{d}u}{\Psi(u)}=\infty$. By Corollary 3.3, almost surely $e^{-\gamma t}\hat{X}_t(x)\underset{t\to\infty}{\longrightarrow}\hat{W}(x)$ for all $x\geq 0$, where $\hat{W}$ is the inverse of a subordinator W with Laplace exponent*

$$\kappa(\theta)=\gamma\log(1+\theta)=\int_0^\infty(1-e^{-\theta x})\gamma\frac{e^{-x}}{x}\mathrm{d}x.$$

The limiting process $\hat{W}$ is therefore an inverse Gamma subordinator.

The next observation ensures that the choice of the parameter λ is arbitrary in both cases i) and ii) in the sense that a change in λ only affects the limit by a multiplicative factor.

Lemma 3.6. *For any $\lambda' \neq \lambda \in (0,\infty)$ and $x \in (0,\infty)$,*

$$\hat{W}^{\lambda'}(x) = c_{\lambda',\lambda}\hat{W}^{\lambda}(x) \text{ almost surely, with } c_{\lambda',\lambda} = e^{\Psi'(0+)\int_\lambda^{\lambda'}\frac{\mathrm{d}u}{\Psi(u)}}.$$

The process $(\hat{W}^\lambda(x), x \geq 0)$ can be interpreted as follows. Define a random equivalence relation $\mathscr{A}$ on $(0,\infty)$ via

$$x \overset{\mathscr{A}}{\sim} y \text{ if and only if } \hat{W}^\lambda(x) = \hat{W}^\lambda(y).$$

This induces a random partition[3] of the set $(0,\infty)$ into open intervals of constancy of $\hat{W}^\lambda$. A simple application of Lemma 3.6 ensures that this partition does not depend on λ. By definition, the subintervals of the partition $\mathscr{A}$ are made of individuals whose ancestral lineages have the same asymptotic behaviour. These subintervals correspond to the jump intervals of W^λ, the subordinator obtained as the right-inverse of $\hat{W}^\lambda$, that is to say

$$\mathscr{A} = \left\{\left(W^\lambda(x-), W^\lambda(x)\right); x > 0\right\} \text{ a.s.}$$

In other words the families in $\mathscr{A}$ are separated by points $x_i, i \in I$, in the support $\mathscr{S}$ of the associated Stieltjes measure $\mathrm{d}\hat{W}^\lambda$, $\mathscr{S} := \overline{\{W^\lambda(x) : x \geq 0\}}$.

The next theorem states that $\mathscr{A}$ corresponds actually to the families of current individuals having a common ancestor.

Theorem 3.7. *For any $x, y \in (0,\infty)$,*

$$x \overset{\mathscr{A}}{\sim} y \text{ if and only if } \hat{X}_t(x) = \hat{X}_t(y) \text{ for some } t \geq 0.$$

Theorem 3.1 and Theorem 3.7 complete the results obtained under Grey's condition, in Foucart et al. [FMM19, Sections 3, 4 and 5.3], on the long-term behaviour of the ancestral lineages, as well as on the representation of the ancestral partition when Grey's condition is not in force. As stated in Theorem 3.1, there are two separate cases to treat according whether Grey's condition holds or not.

When $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} < \infty$ (Grey's condition), the process $(\hat{W}^\lambda(x), x \geq 0)$ is the inverse of a compound Poisson process for any $\lambda \in (0,\infty]$. By taking $\lambda = \infty$, the latter has for jump measure the probability law ν_∞ and the partition $\mathscr{A}$ is thus constituted of i.i.d. families with lengths of law ν_∞, i.e. $\mathscr{A}$ takes the form of a consecutive partition into intervals:

$$\mathscr{A} = \big((0,x_1),(x_1,x_2),\dots\big) \text{ a.s.,}$$

where $(x_i, i \geq 1)$ is a random renewal process with jump law ν_∞. The following figure provides a schematic representation of the families, their lineages and the process $\hat{W}^\infty$, under Grey's condition.

When $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} = \infty$, the description is more involved since the process $(\hat{W}^\lambda(x), x \geq 0)$ has singular continuous paths and any fixed subinterval of $(0,\infty)$ of finite length contains infinitely many small families with positive probability. Recall that $\mathscr{S}$ is the support of the random singular measure $\mathrm{d}\hat{W}^\lambda$.

[3] up to a negligible set

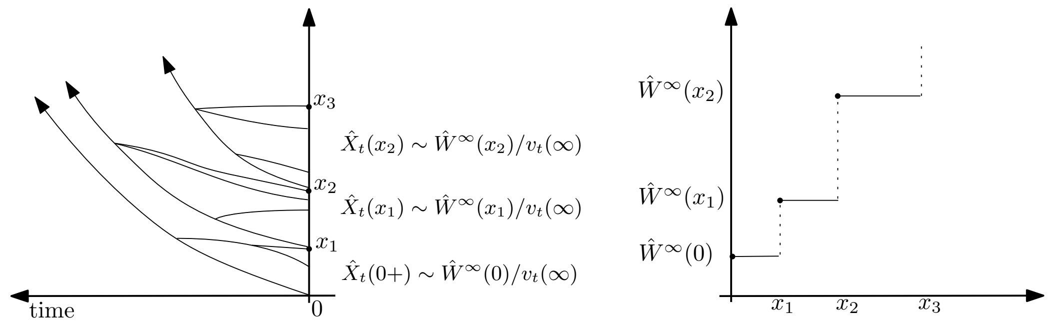

FIGURE 2. Schematic representation of ancestral families under Grey's condition

When Ψ is not the branching mechanism of the subcritical Feller diffusion, i.e. $\pi \not\equiv 0$, coalescences between the ancestral lineages inside each family are possibly multiple and can be described using the notion of consecutive coalescents, see [FMM19, Section 5].

Proposition 3.8. *Set* $\Psi'(\infty) := \lim_{u\to\infty} \frac{\Psi(u)}{u} \in (0,\infty]$. *For any* $x > 0$, *the Hausdorff dimension of* $\mathscr{S} \cap [0,x]$ *is*

$$\dim_H(\mathscr{S} \cap [0,x]) = \frac{\Psi'(0+)}{\Psi'(\infty)} \in [0,1) \ \textit{a.s.} \tag{3.15}$$

From (2.5), one sees that $\Psi'(\infty) = \frac{\sigma^2}{2} \cdot \infty + \gamma + \int_0^\infty x\pi(\mathrm{d}x) \in (0,\infty]$. In the case of a CSBP with unbounded variation, namely with $\sigma > 0$ or $\int_0^1 x\pi(\mathrm{d}x) = \infty$, one has $\Psi'(\infty) = \infty$ and the Hausdorff dimension of $\mathscr{S}$ is zero. In the bounded variation case, (3.15) can be rewritten as

$$\dim_H(\mathscr{S} \cap [0,x]) = \frac{\gamma}{\gamma + \int_0^\infty h\pi(\mathrm{d}h)} \ \text{a.s.}$$

Remark 3.9. If one denotes by $\mathrm{PPP}_\lambda := \sum_{i\in I} \delta_{(a_i^\lambda, \Delta_i^\lambda)}$ the Poisson point process associated to the subordinator W^λ, the atoms $(\Delta_i^\lambda, i \in I)$ are by definition the sizes of the different families in $\mathscr{A}$ and the atoms of jump times $(a_i^\lambda, i \in I)$ are related to the rate of escape of the ancestral lineages and can be thought as some ancestral types. Let $\epsilon > 0$. The restriction of PPP_λ to the set $(0,\infty) \times (\epsilon,\infty)$ takes the form $\mathrm{PPP}_{\lambda|(0,\infty)\times(\epsilon,\infty)} = \sum_{j=1}^\infty \delta_{(\mathbb{e}_j^\epsilon, D_j^\epsilon)}$, where $(\mathbb{e}_j^\epsilon, j \geq 1)$ is a sequence of i.i.d. exponential random variables with parameter $\bar{\nu}_\lambda(\epsilon) := \nu_\lambda\big((\epsilon,\infty)\big)$ and $(D_j^\epsilon, j \geq 0)$ are i.i.d. random variables with law $\frac{\nu_\lambda(\mathrm{d}x)}{\bar{\nu}_\lambda(\epsilon)} \mathbb{1}_{(\epsilon,\infty)}(x)$. The exponential random variables allow one to distinguish families between each others. If $\mathbb{e}_i^\epsilon$ is larger than $\mathbb{e}_j^\epsilon$, then the ancestral lineage of the j^{th} family with size larger than ϵ, diverges slower than that of the i^{th}. Note that when Grey's condition holds, the characteristic measure of the Poisson point process PPP_λ is finite and one can take $\epsilon = 0$. When Grey's condition fails, one needs first to restrict ourselves to families of size greater than $\epsilon > 0$ in order to be able to rank the atoms of times of PPP_λ. In the schematic representation given in Figure 2, the divergence of the ancestral lineage of the family (x_2, x_3) is faster than those of (x_1, x_2) and $(0, x_1)$.

Example 3.10. *Let* Ψ *be the branching mechanism with drift* $\gamma = 1$ *and Lévy measure* $\pi(\mathrm{d}x) = x^{-\alpha-1}e^{-x}\mathrm{d}x$ *with* $\alpha \in (0,2)$. *Then by a Tauberian theorem, see e.g. Feller's book*

[Fel71, Chapter XIII.5],
$$\frac{\Psi(x)}{x} \underset{x\to\infty}{\sim} \int_0^\infty (1-e^{-xh})h\bar{\pi}(h)\mathrm{d}h \underset{x\to\infty}{\sim} c_\alpha x^{\alpha-1},$$
where for all $h>0$, $\bar{\pi}(h) := \pi((h,\infty))$ and c_α is a strictly positive constant.

i) *If $\alpha \in (1,2)$, then $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} < \infty$, hence $\Psi'(\infty)=\infty$, and*
$$\mathscr{S} \text{ is a discrete set, thus } \dim_H(\mathscr{S}\cap[0,x]) = 0 \text{ a.s. for all } x>0.$$
ii) *If $\alpha = 1$ (Neveu case), then $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} = \infty$ and $\Psi'(\infty)=\infty$, hence*
$$\mathscr{S} \text{ is not a discrete set, but } \dim_H(\mathscr{S}\cap[0,x]) = 0 \text{ a.s. for all } x>0.$$
iii) *If $\alpha < 1$, $\Psi'(\infty) < \infty$, hence $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} = \infty$ and*
$$\mathscr{S} \text{ is not a discrete set, and } \dim_H(\mathscr{S}\cap[0,x]) = \frac{1}{1+\Gamma(1-\alpha)} \text{ a.s. for all } x>0.$$

In general, inverse subordinators do not have the Markov property. The joint density of the finite-dimensional marginals of $(\hat{W}^\lambda(x), x\geq 0)$ are thus rather involved. We refer to the works of Lageras [Lag05] and Veillette and Taqqu [VT10] for information on inverse subordinators.

The following proposition is a side result on the one-dimensional laws of the limiting process $(\hat{W}^\lambda(x), x\geq 0)$. Recall that ν_λ denotes the Lévy measure of the subordinator $(W^\lambda(x), x\geq 0)$ and $\bar{\nu}_\lambda$ is its tail: for any $x\geq 0$, $\bar{\nu}_\lambda(x) := \nu_\lambda\big((x,\infty)\big)$.

Proposition 3.11. *The law of $\hat{W}^\lambda(x)$ admits the density g_x^λ defined on $(0,\infty)$ by*
$$g_x^\lambda(u) := \int_0^x \bar{\nu}_\lambda(x-z)\mathbb{P}(W^\lambda(u)\in \mathrm{d}z). \tag{3.16}$$
When $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} < \infty$, $\hat{W}^\infty(x)$ has density g_x^∞ :
$$g_x^\infty(u) := e^{-u}\sum_{n=0}^\infty \frac{u^n}{n!}\int_0^x \bar{\nu}_\infty(x-z)\nu_\infty^{\star n}(\mathrm{d}z). \tag{3.17}$$

4. Proofs

The most demanding part in the proof of Theorem 3.1 lies in the almost-sure convergence. The convergence in law will be established from a direct argument involving the subordinators $(X_{-t,0}(x), x\geq 0)$ rather than their inverse, we refer to the forthcoming Lemma 4.14. In order to show the almost-sure convergence, we first establish a general result of independent interest relating θ-invariant functions of stochastically monotone processes, with θ-invariant measures of their dual processes, see Section 4.1. We then apply this result in the setting of CSBPs. The asymptotics of a certain θ-invariant function for the dual process $\hat{X}$ is studied. It enables to find a new martingale associated to the ancestral lineage process. We shall see how the renormalisation $v_t(\lambda)$ appears in this martingale and deduce the almost sure convergence of the process $(v_t(\lambda)\hat{X}_t(x), t\geq 0)$ started from a fixed value x, see Lemma 4.10. We study the associated limiting process in x and show that it satisfies the properties stated in Theorem 3.1, see Lemma 4.11.

Theorem 3.7 will be established with the help of some previous results obtained in [FMM19]. Proposition 3.8 will be a consequence of Theorem 3.1. The proof of Theorem 3.1

is divided into several lemmas. The first are needed to show the almost sure convergence towards some positive random variable $\hat{W}^\lambda(x)$.

4.1. **Invariant functions of stochastically monotone Markov processes.** In this section, we consider a "general" standard Markov process $X := (X_t, t \geq 0)$ with state space $[0,\infty)$, and denote by $(X_t(y), t \geq 0)$ the process started from $y \in [0,\infty)$. Recall that the process X is said to be stochastically monotone if for any $t \geq 0$ and $x \in [0,\infty)$, the map $y \mapsto \mathbb{P}(X_t(y) \geq x)$ is non-decreasing. Siegmund [Sie76] has established that if the process X is stochastically monotone, non-explosive or with boundary ∞ absorbing, and that for any fixed t and z, the map $y \mapsto \mathbb{P}(X_t(y) \geq z)$ is right-continuous then there exists a unique Markov process $\hat{X}$, the so-called Siegmund dual process, such that for any t and x, y

$$\mathbb{P}(X_t(y) \geq x) = \mathbb{P}(\hat{X}_t(x) \leq y). \tag{4.18}$$

The latter identity can be rewritten as

$$\mathbb{P}\big(\hat{X}_t(x) > y\big) = \mathbb{P}\big(x > X_t(y)\big). \tag{4.19}$$

Our first result shows how to find fundamental martingales for the Siegmund dual process $(\hat{X}_t(x), t \geq 0)$ of any stochastically monotone Markov process $(X_t(x), t \geq 0)$. Recall $\hat{T}_y$ defined in (2.11).

Theorem 4.1 (Invariant functions of $\hat{X}$)**.** *Let $(P_t, t \geq 0)$ be the semigroup of the process $(X_t, t \geq 0)$. Let $\theta \in \mathbb{R}$. If μ_θ is a positive Borel measure on $(0,\infty)$ satisfying for any $t \geq 0$, $\mu_\theta P_t = e^{\theta t}\mu_\theta$, then the functions $x \mapsto \mu_\theta([0,x))$ and $x \mapsto \mu_\theta((x,\infty))$, provided they are well-defined, are θ-invariant functions, namely functions f_θ such that for any $t \geq 0$ and $x \in [0,\infty)$,*

$$\mathbb{E}[f_\theta(\hat{X}_t(x))] = e^{\theta t} f_\theta(x),$$

so that

$$\big(e^{-\theta t} f_\theta(\hat{X}_t(x)), t \geq 0\big) \text{ is a martingale.} \tag{4.20}$$

In particular, if the process $(\hat{X}_t, t \geq 0)$ has no positive jumps and μ_θ is finite on $[0,x)$ for all $x > 0$, then $f_\theta : x \mapsto \mu_\theta([0,x))$ is a well-defined increasing and left-continuous function, and for all $y \geq x \geq 0$,

$$\mathbb{E}_x[e^{-\theta \hat{T}_y}] = \frac{\mu_\theta([0,x))}{\mu_\theta([0,y))}. \tag{4.21}$$

Remark 4.2. A measure μ_θ satisfying $\mu_\theta P_t = e^{\theta t}\mu_\theta$ is sometimes called an *eigen-measure* or a θ-invariant measure.

Remark 4.3. Let $\mathcal{L}$ be the generator of the process $(X_t, t \geq 0)$. The Kolmogorov forward equation entails that the condition $\mu_\theta P_t = e^{\theta t}\mu_\theta$ for all $t \geq 0$, is equivalent to $\mu_\theta \mathcal{L} = \theta \mu_\theta$ where $\mu_\theta \mathcal{L}$ is by definition the measure such that $\langle \mu_\theta \mathcal{L}, f\rangle := \int \mathcal{L}f(x)\mu_\theta(\mathrm{d}x)$ for any function $f \in C^2_b((0,\infty))$.

Proof. Set $f_\theta(x)=\mu_\theta([0,x))$ for all $x>0$. For any $x\in(0,\infty)$ and any $t\geq 0$,

$$\begin{aligned}\hat{P}_t f_\theta(x)=\mathbb{E}\left[f_\theta\big(\hat{X}_t(x)\big)\right]&=\mathbb{E}\left[\int \mathbb{1}_{\{\hat{X}_t(x)>y\}}\mu_\theta(\mathrm{d}y)\right]\\ &=\mathbb{E}\left[\int \mathbb{1}_{\{x>X_t(y)\}}\mu_\theta(\mathrm{d}y)\right] \text{ by the duality relation (4.19)}\\ &=\mu_\theta P_t\big([0,x)\big)=e^{\theta t}\mu_\theta\big([0,x)\big)=e^{\theta t}f_\theta(x).\end{aligned}$$

The martingale property (4.20) follows readily from the Markov property. Note that the map $f_\theta : x\mapsto \mu_\theta([0,x))$ is left-continuous. We now apply the bounded optional stopping time theorem at time $t\wedge \hat{T}_y$:

$$\mathbb{E}\big[e^{-\theta t\wedge \hat{T}_y}f_\theta\big(\hat{X}_{t\wedge\hat{T}_y}(x)\big)\big]=f_\theta(x).$$

Since f_θ is non-decreasing and $\hat{X}_{t\wedge\hat{T}_y}(x)\leq y$ a.s, one has for any $t\geq 0$, $f_\theta\big(\hat{X}_{t\wedge\hat{T}_y}(x)\big)\leq f_\theta(y)$. On the event $\{\hat{T}_y<\infty\}$, the left-continuity of f_θ and the absence of negative jumps in the process $(\hat{X}_t,t\geq 0)$ ensure that $f_\theta\big(\hat{X}_{t\wedge\hat{T}_y}(x)\big)\underset{t\to\infty}{\longrightarrow} f_\theta(y)$. This yields

$$f_\theta(x)=\lim_{t\to\infty}\mathbb{E}\big[e^{-\theta t\wedge\hat{T}_y}f_\theta\big(\hat{X}_{t\wedge\hat{T}_y}(x)\big)\big]=\mathbb{E}\big[e^{-\theta\hat{T}_y}f_\theta(y)\mathbb{1}_{\{\hat{T}_y<\infty\}}\big],$$

which provides the identity (4.21). □

Remark 4.4. Theorem 4.1 holds in general for any stochastically monotone Markov process. In particular, the process is not required to have one-sided jumps. The state space $[0,\infty]$ could also be replaced by a more general nice ordered state space.

Remark 4.5. Let $\hat{\mathcal{L}}$ denote the generator of the process $\hat{X}$. An invariant function f_θ for the semigroup of $\hat{X}$ can be thought as a solution to the equation $\hat{\mathcal{L}}f_\theta=\theta f_\theta$. However, when the process has jumps, $\hat{\mathcal{L}}$ is an integro-differential operator and no general theory allows one for identifying solutions of this equation. Lemma 4.1 reveals that for stochastically monotone processes finding a θ-invariant function corresponds to finding a θ-invariant measure for the dual process. This is reminiscent to a result of Cox and Rösler [CR84].

4.2. **Application to CSBPs and proof of Theorem 3.1.** Recall the definition of $(\hat{X}_t(x),t\geq 0)$ as the right-continuous inverse of $(X_{-t,0}(x),t\geq 0)$, Lemma 2.4 and the duality relation (2.13). This relationship matches with (4.19) and we will be able to apply Theorem 4.1.

Recall the action (2.4) of the generator $\mathcal{L}$ on exponential functions. We now look for the θ-invariant measures μ_θ in our setting and their Laplace transforms explicitly in terms of Ψ. The following Lemma holds for general branching mechanism (i.e not necessarily subcritical).

Lemma 4.6. *For any $\theta>0$, the map $c_\theta : q\mapsto e^{-\theta\int_1^q \frac{\mathrm{d}u}{\Psi(u)}}$ is the Laplace transform of a Borel measure μ_θ on $[0,\infty)$. Moreover, the measure μ_θ is θ-invariant for the semigroup $(P_t,t\geq 0)$ of the CSBP $(X_t,t\geq 0)$.*

Proof. Recall that $\mathcal{L}$ denotes the generator of the CSBP $(X_t(x),t\geq 0)$. Let $\theta\geq 0$. It is easily checked from the expression of c_θ that $(-1)^n c_\theta^{(n)}\geq 0$ on $(0,\infty)$. Bernstein theorem, see e.g. [Wid41, Theorem 12.b, page 161], guarantees that there exists a certain Borel measure μ_θ (possibly infinite) on $[0,\infty)$ such that $c_\theta(q)=\int_{(0,\infty)} e^{-qx}\mu_\theta(\mathrm{d}x)$ for all $q>0$. We now check that the measure μ_θ solves $\mu_\theta\mathcal{L}=\theta\mu_\theta$. Let $q>0$, recall that $\lambda>0$ is fixed. Observe first that c_θ satisfies the equation

$$-\Psi(q)c_\theta'(q)=\theta c_\theta(q),\quad c_\theta(1)=1. \tag{4.22}$$

Recall $e_q : x\mapsto e^{-qx}$ for any $x,q\geq 0$. Since the linear span of exponential functions is a core for the generator $\mathcal{L}$, it is enough to verify that

$$\langle \mu_\theta\mathcal{L}, e_q\rangle := \int_0^\infty \mathcal{L}e_q(x)\mu_\theta(\mathrm{d}x)=\langle\theta\mu_\theta,e_q\rangle=\theta\int_0^\infty e^{-qx}\mu_\theta(\mathrm{d}x). \tag{4.23}$$

One has on the other hand $\mathcal{L}e_q(x)=\Psi(q)xe_q(x)$ and (4.23) is equivalent to

$$\Psi(q)\int_0^\infty xe^{-qx}\mu_\theta(\mathrm{d}x)=\theta\int_0^\infty e^{-qx}\mu_\theta(\mathrm{d}x), \tag{4.24}$$

which holds true by using (4.22). □

Remark 4.7. In the subcritical or critical cases, the map $q\mapsto c_\theta(q)$ is completely monotone on $(0,\infty)$ and not defined at 0. This entails that the measure μ_θ is infinite. In the supercritical case, c_θ is completely monotone *and* well-defined and right-continuous at 0, in this case the measure μ_θ is finite.

According to Theorem 4.1, the map $f_\theta : x\mapsto \mu_\theta([0,x))$ is a θ-invariant function for $(\hat{X}_t, t\geq 0)$. The following simple calculation provides an expression of the Laplace transform of f_θ. For any $q>0$,

$$\xi_\theta(q):=\int_0^\infty f_\theta(y)e^{-qy}\mathrm{d}y=\int_0^\infty\int_0^\infty \mathbb{1}_{\{u<y\}}e^{-qy}\mu_\theta(\mathrm{d}u)\mathrm{d}y=\frac{1}{q}c_\theta(q). \tag{4.25}$$

Inverting ξ_θ in order to find f_θ does not seem to be feasible in a general setting, however we shall see in the next lemma that ξ_θ has regular variation properties at 0, Tauberian theorems will then allow us to find an equivalent at ∞ of the function f_θ and hence enable us to investigate more precisely the martingale $(e^{-\theta t}f_\theta(\hat{X}_t(x)), t\geq 0)$.

Lemma 4.8. *Assume* $\Psi'(0+)\neq 0$. *The map* $R: q\mapsto e^{-\int_1^q \frac{\mathrm{d}u}{\Psi(u)}}$ *is regularly varying at* 0 *with index* $-1/\Psi'(0+)$. *In particular it takes the form* $R(q)=q^{-\frac{1}{\Psi'(0+)}}L_1(1/q)$, *where* L_1 *is a slowly varying function at* ∞. *Moreover, for any* $\theta>-\Psi'(0+)$,

$$f_\theta(y)\underset{y\to\infty}{\sim} y^{\frac{\theta}{\Psi'(0+)}}\frac{1}{\Gamma\left(1+\frac{\theta}{\Psi'(0+)}\right)}L_1(y)^\theta=\frac{1}{\Gamma\left(1+\frac{\theta}{\Psi'(0+)}\right)}R(1/y)^\theta. \tag{4.26}$$

Proof. For any $q>0$,

$$\begin{aligned}\int_q^1\frac{\mathrm{d}u}{\Psi(u)}&=\int_q^1\left(\frac{1}{\Psi(u)}-\frac{1}{\Psi'(0+)u}\right)\mathrm{d}u+\int_q^1\frac{\mathrm{d}u}{\Psi'(0+)u}\\&=\int_1^{1/q}\left(\frac{1}{u^2\Psi(1/u)}-\frac{1}{\Psi'(0+)u}\right)\mathrm{d}u-\frac{1}{\Psi'(0+)}\log(q).\end{aligned}$$

Set $\epsilon(u) = \frac{1}{u\Psi(1/u)} - \frac{1}{\Psi'(0+)}$ for $u > 0$ and notice that $\epsilon(u) \underset{u\to\infty}{\longrightarrow} 0$. One has

$$R(q) = q^{-\frac{1}{\Psi'(0+)}} \exp\left(\int_1^{1/q} \frac{\epsilon(u)}{u}\mathrm{d}u\right)$$

and by Karamata's representation theorem, see e.g. Bingham et al. [BGT87, Theorem 1.3.1], we see that $L_1(x) := \exp\left(\int_1^x \frac{\epsilon(u)}{u}\mathrm{d}u\right)$ is slowly varying at ∞. Recall $c_\theta(q) = e^{-\theta\int_1^q \frac{\mathrm{d}u}{\Psi(u)}}$. By (4.25), $\xi_\theta : q \mapsto \frac{c_\theta(q)}{q}$ is regularly varying at 0 with index $\rho = -1 - \frac{\theta}{\Psi'(0+)}$. The Tauberian theorem with monotone density, see e.g. [Fel71, Chapter XIII.5, Theorem 4], provides (4.26). □

From now on, we focus on the subcritical case $\Psi'(0+) > 0$. The two next lemmas are shown by adapting arguments of Pakes [Pak17, Theorem 10] and Barbour [Bar75] to the setting of the continuous-state space. The first one provides an almost sure limit theorem for the first passage time above level y, $\hat{T}^x_y$, when y goes to ∞.

Lemma 4.9. *For any $x > 0$, almost-surely*

$$\hat{T}^x_y - \int_{1/y}^1 \frac{\mathrm{d}u}{\Psi(u)} \underset{y\to\infty}{\longrightarrow} S(x),$$

where $S(x)$ is a finite random variable with Laplace transform

$$\mathbb{E}[e^{-\theta S(x)}] = \Gamma\left(1 + \frac{\theta}{\Psi'(0+)}\right) f_\theta(x) \text{ for all } \theta > -\Psi'(0+). \tag{4.27}$$

Proof. We establish first a convergence in law. By the asymptotic equivalence (4.26), for any $\theta > -\Psi'(0+)$,

$$\mathbb{E}[e^{-\theta(\hat{T}^x_y - \int_{1/y}^1 \frac{\mathrm{d}u}{\Psi(u)})}] \underset{y\to\infty}{\longrightarrow} \Gamma\left(1 + \frac{\theta}{\Psi'(0+)}\right) f_\theta(x).$$

One obtains by applying a continuity theorem for the moment generating function, see [Cur42, Theorem 3], that as y goes to ∞,

$$\hat{T}^x_y - \int_{1/y}^1 \frac{\mathrm{d}u}{\Psi(u)} \xrightarrow{\mathcal{L}} S(x), \tag{4.28}$$

where $S(x)$ has Laplace transform (4.27). We now show the almost sure convergence. Let $(y_n, n \geq 0)$ be an increasing sequence such that $y_n \underset{n\to\infty}{\longrightarrow} \infty$ and with $y_0 := x$. Write

$$\hat{T}^x_{y_n} - \int_{1/y_n}^{1/x} \frac{\mathrm{d}u}{\Psi(u)} = \sum_{k=0}^{n} \left(\hat{T}^x_{y_{k+1}} - \hat{T}^x_{y_k} - \int_{1/y_{k+1}}^{1/y_k} \frac{\mathrm{d}u}{\Psi(u)}\right).$$

Since the Markov process $\hat{X}$ has no positive jumps, the summands are independent and by (4.28), the series on the right-hand side converges in law, one can apply [Chu68, Theorem 9.5.5], which ensures that the series actually converges almost-surely. We finally get that

$$\hat{T}^x_y - \int_{1/y}^1 \frac{\mathrm{d}u}{\Psi(u)} \underset{y\to\infty}{\longrightarrow} S(x) \text{ a.s.}$$

□

We now deduce the convergence of the process $(v_t(\lambda)\hat{X}_t(x), t \geq 0)$ for fixed x and a representation of its limit $\hat{W}^\lambda(x)$.

Lemma 4.10. *Let $\lambda > 0$. For any $x > 0$, the following almost sure convergence holds:*

$$v_t(\lambda)\hat{X}_t(x) \underset{t\to\infty}{\longrightarrow} \hat{W}^\lambda(x) := c(\lambda)e^{-\Psi'(0+)S(x)}, \tag{4.29}$$

where $c(\lambda)$ is a constant independent of x.

Proof. We first establish the almost sure convergence towards some random variable $\hat{W}^\lambda(x)$. By applying Theorem 4.1, the process $(e^{-t}f_1(\hat{X}_t(x)), t \geq 0)$ is a positive martingale. Therefore, the latter converges almost surely towards some random variable $Z(x)$. Recall Lemma 4.8. Set $\beta(1,\lambda) := \frac{\lambda\xi_1(\lambda)}{\Gamma(1+1/\Psi'(0+))}$ and $R_\lambda(1/y) := \exp\left(\int_{1/y}^{\lambda} \frac{du}{\Psi(u)}\right) = \exp\left(\int_{1}^{\lambda} \frac{\mathrm{d}u}{\Psi(u)}\right) R(1/y)$. Recall that $\hat{X}_t(x) \underset{t\to\infty}{\longrightarrow} \infty$ a.s, see Proposition 2.3. By Lemma 4.8,

$$e^{-t}f_1\big(\hat{X}_t(x)\big) \underset{t\to\infty}{\sim} e^{-t}\beta(1,\lambda)R_\lambda\big(1/\hat{X}_t(x)\big) \underset{t\to\infty}{\longrightarrow} Z(x) \text{ a.s.}$$

Hence

$$R_\lambda\big(1/\hat{X}_t(x)\big) \underset{t\to\infty}{\sim} \frac{e^t Z(x)}{\beta(1,\lambda)}. \tag{4.30}$$

Now, using the fact that R_λ is non-decreasing and regularly varying at 0 with index $-\frac{1}{\Psi'(0+)}$, we see that R_λ^{-1} is regularly varying at ∞ with index $-\Psi'(0+)$. Taking R_λ^{-1} in the asymptotic equivalence (4.30) yields

$$\frac{1}{\hat{X}_t(x)} \underset{t\to\infty}{\sim} R_\lambda^{-1}\left(\frac{e^t Z(x)}{\beta(1,\lambda)}\right) \underset{t\to\infty}{\sim} \left(\frac{Z(x)}{\beta(1,\lambda)}\right)^{-\Psi'(0+)} R_\lambda^{-1}(e^t).$$

By the definition of $\lambda \mapsto v_t(\lambda)$, see (2.7), one has $R_\lambda\big(v_t(\lambda)\big) = e^t$. This allows us to conclude that

$$v_t(\lambda)\hat{X}_t(x) \underset{t\to\infty}{\longrightarrow} \hat{W}^\lambda(x) := \big(Z(x)/\beta(1,\lambda)\big)^{\Psi'(0+)} \text{ a.s.}$$

We now explain the relation between $\hat{W}^\lambda(x)$ and the random variable $S(x)$ introduced in the previous lemma. Denote by $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration of $\hat{X}$. Let $(M_t, t \geq 0)$ be the martingale defined by $M_t := \mathbb{E}[e^{-S(x)}|\mathcal{F}_t]$ for any $t \geq 0$. Note that $M_t \underset{t\to\infty}{\longrightarrow} e^{-S(x)}$ almost surely. Moreover by Lemma 4.9, and Lebesgue's theorem, for any $t \geq 0$,

$$M_t = \lim_{y\to\infty} \mathbb{E}\left[e^{-\left(\hat{T}_y^x - \int_{1/y}^{1} \frac{\mathrm{d}u}{\Psi(u)}\right)}|\mathcal{F}_t\right].$$

For any fixed y, conditional on $\{\hat{T}_y^x > t\}$, we have that $\hat{T}_y^x = \hat{T}_y^x \circ \theta_t + t$ with $\hat{T}_y^x \circ \theta_t := \inf\{s > 0 : \hat{X}_{s+t}(x) > y\}$. By the Markov property, we get almost surely

$$\mathbb{E}[e^{-\hat{T}_y^x}|\mathcal{F}_t] = \frac{f_1\left(\hat{X}_{t\wedge\hat{T}_y^x}(x)\right)}{f_1(y)} e^{-t\wedge\hat{T}_y^x}.$$

Therefore, almost surely

$$\begin{aligned} M_t &= \lim_{y\to\infty} \frac{e^{\int_{1/y}^{1}\frac{\mathrm{d}u}{\Psi(u)}}}{f_1(y)} f_1\left(\hat{X}_{t\wedge \hat{T}^x_y}(x)\right) e^{-t\wedge \hat{T}^x_y} \\ &= \Gamma\left(1+\frac{1}{\Psi'(0+)}\right) f_1(\hat{X}_t(x))e^{-t}. \end{aligned}$$

Hence,

$$Z(x) := \lim_{t\to\infty} e^{-t} f_1(\hat{X}_t(x)) = \frac{1}{\Gamma\left(1+\frac{1}{\Psi'(0+)}\right)} e^{-S(x)} > 0 \text{ a.s.},$$

and we have $\hat{W}^\lambda(x) = c(\lambda)e^{-\Psi'(0+)S(x)}$ a.s. □

The almost sure convergence in Lemma 4.10 holds for fixed x. We now further investigate the limiting process in the variable x and establish a stronger convergence.

Lemma 4.11. *The process $(\hat{W}^\lambda(x), x \geq 0)$ admits a non-decreasing right-continuous modification. Moreover, setting $J^\lambda := \{x > 0 : \hat{W}^\lambda(x) > \hat{W}^\lambda(x-)\}$, one has almost surely,*

$$\forall x \notin J^\lambda,\ v_t(\lambda)\hat{X}_t(x) \underset{t\to\infty}{\longrightarrow} \hat{W}^\lambda(x). \tag{4.31}$$

Proof. For technical reasons it will be easier to establish the result first with a left-continuous limiting random process instead of the targeted right-continuous one. Once a version with left-continuous paths and right limits is constructed, one can easily consider the associated right-continuous version of it. Recall that by Lemma 4.10, for any fixed x, $v_t(\lambda)\hat{X}_t(x)$ converges almost surely as t goes to ∞ towards a random variable denoted by $\hat{W}^\lambda(x)$. Consider the almost sure event Ω_1 on which all random variables $(\hat{W}^\lambda(q), q \in \mathbb{Q}_+)$ are defined. Since for any $t \geq 0$, the process $(\hat{X}_t(x), x \geq 0)$ is non-decreasing then for any rational numbers $q' \geq q \geq 0$, one has $\hat{W}^\lambda(q') \geq \hat{W}^\lambda(q)$. We work deterministically on Ω_1 and define for all $x \geq 0$,

$$\tilde{\hat{W}}^\lambda(x) := \lim_{\substack{q\uparrow x,\\ q\in\mathbb{Q}_+}} \hat{W}^\lambda(q).$$

The process $(\tilde{\hat{W}}^\lambda(x), x \geq 0)$ is well-defined on Ω_1 and we define it as the null process on $\Omega \setminus \Omega_1$. By construction, $(\tilde{\hat{W}}^\lambda(x), x \geq 0)$ is left-continuous. It has right limits since it is non-decreasing. We now show that for any $x \geq 0$, $\mathbb{P}(\hat{W}^\lambda(x) = \tilde{\hat{W}}^\lambda(x)) = 1$. By the identity (4.29), on the event Ω_1, we have that for all $q \in \mathbb{Q}_+$ such that $q < x$

$$\hat{W}^\lambda(q) = \hat{W}^\lambda(x)e^{-\Psi'(0+)(S(q)-S(x))} \text{ a.s.} \tag{4.32}$$

We now verify that $S(q) \underset{\substack{q\to x\\ q<x}}{\longrightarrow} S(x)$ a.s. Since almost surely for all $x \leq q \leq q'$, $\hat{X}_t(x) \leq \hat{X}_t(q) \leq \hat{X}_t(q')$, then $\hat{T}^{q'}_y \leq \hat{T}^q_y \leq \hat{T}^x_y$ and similarly, almost surely

$$S(q') \leq S(q) \leq S(x).$$

Hence $(S(q), q > 0)$ is decreasing. Recall the Laplace transform of $S(q)$ given in (4.27), since the function $f_\theta : x \mapsto \mu_\theta([0, x))$ is left-continuous, we have that

$$\mathbb{E}[e^{-\theta S(q)}] \underset{\substack{q\to x\\ q<x}}{\longrightarrow} \mathbb{E}[e^{-\theta S(x)}].$$

Finally we see that $S(q) \underset{\substack{q\to x\\ q<x}}{\longrightarrow} S(x)$ a.s. and by the identity (4.32), $\hat{W}^\lambda(q) \underset{q\uparrow x, q\in\mathbb{Q}}{\longrightarrow} \hat{W}^\lambda(x)$, hence $\hat{W}^\lambda(x) = \tilde{\hat{W}}^\lambda(x)$ a.s. and $\tilde{\hat{W}}^\lambda$ is a left-continuous version of the family of random variables $(\hat{W}^\lambda(x), x \geq 0)$. If we now define on Ω_1, simultaneously for all x, the process $(\hat{W}^\lambda(x), x \geq 0)$ by setting $\hat{W}^\lambda(x) := \tilde{\hat{W}}^\lambda(x+)$, then the process $(\hat{W}^\lambda(x), x \geq 0)$ is right-continuous. The first statement of the lemma is established. It remains to see that the almost sure pointwise convergence holds outside the set of jumps J^λ. Almost surely for all $x \notin J^\lambda$, one can choose two rational numbers q and q' such that $q' < x < q$. Since $\hat{X}_t(q') \leq \hat{X}_t(x) \leq \hat{X}_t(q)$ for all t, one has

$$\tilde{\hat{W}}^\lambda(q') \leq \liminf_{t\to\infty} v_t(\lambda)\hat{X}_t(x) \leq \limsup_{t\to\infty} v_t(\lambda)\hat{X}_t(x) \leq \tilde{\hat{W}}^\lambda(q).$$

Since $\tilde{\hat{W}}^\lambda$ has left-continuous paths with right limits and $x \notin J^\lambda$, both sides of the inequalities above converge towards the same value $\tilde{\hat{W}}^\lambda(x)$ when $q' \uparrow x$ and $q \downarrow x$. By definition of $\hat{W}^\lambda(x)$, since $x \notin J^\lambda$, $\hat{W}^\lambda(x) = \tilde{\hat{W}}^\lambda(x)$. This allows us to claim (4.31). □

From now on we work with the right-continuous version of $\hat{W}^\lambda$. The next lemma sheds some light on the role of the parameter λ and provides Lemma 3.6.

Lemma 4.12. *For any $\lambda > 0$ and $\lambda' > 0$,*

$$\frac{v_t(\lambda)}{v_t(\lambda')} \underset{t\to\infty}{\longrightarrow} c_{\lambda,\lambda'} := e^{\Psi'(0+)\int_{\lambda'}^{\lambda}\frac{\mathrm{d}u}{\Psi(u)}}. \tag{4.33}$$

Moreover $\hat{W}^\lambda(x) = c_{\lambda,\lambda'}\hat{W}^{\lambda'}(x)$ for all $x \in (0,\infty)$ almost surely.

Proof. Since $\Psi'(0+) \geq 0$, $v_t(\lambda) \underset{t\to\infty}{\longrightarrow} 0$. Moreover, $\lim_{t\to\infty}\uparrow \frac{\Psi(v_t(u))}{v_t(u)} = \Psi'(0+)$. Recall that $\frac{\mathrm{d}}{\mathrm{d}\lambda}v_t(\lambda) = \frac{\Psi(v_t(\lambda))}{\Psi(\lambda)}$. Therefore for any $\lambda \neq \lambda'$,

$$\frac{v_t(\lambda)}{v_t(\lambda')} = \exp\left(\int_{\lambda'}^{\lambda}\frac{\mathrm{d}}{\mathrm{d}u}\log v_t(u)\mathrm{d}u\right) = \exp\left(\int_{\lambda'}^{\lambda}\frac{\Psi(v_t(u))}{v_t(u)}\frac{\mathrm{d}u}{\Psi(u)}\right)$$

and by monotone convergence $\frac{v_t(\lambda)}{v_t(\lambda')} \underset{t\to\infty}{\longrightarrow} \exp\left(\Psi'(0+)\int_{\lambda'}^{\lambda}\frac{\mathrm{d}u}{\Psi(u)}\right)$. We see from Lemma 4.11 that $\hat{W}^\lambda(x) = c_{\lambda,\lambda'}\hat{W}^{\lambda'}(x)$ for all $x \in (0,\infty)$ almost surely. □

Lemma 4.13. *The map $\kappa_\lambda : q \mapsto \mathrm{e}^{-\Psi'(0+)\int_q^\lambda \frac{\mathrm{d}u}{\Psi(u)}}$ is the Laplace exponent of a drift-free subordinator W^λ. Its Lévy measure, denoted by ν_λ, is finite if and only if $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} < \infty$.*

Proof. For any $\theta > 0$ and any $y \in (0,\infty)$, since $X_{-t,0}(z)$ has the same law as $X_{0,t}(z)$ for all z and $t \geq 0$, one gets by (2.6) and by applying Lemma 4.12

$$\mathbb{E}[e^{-\theta X_{-t,0}\left(\frac{y}{v_t(\lambda)}\right)}] = e^{-y\frac{v_t(\theta)}{v_t(\lambda)}} \underset{t\to\infty}{\longrightarrow} e^{-y\kappa_\lambda(\theta)}. \tag{4.34}$$

At any time t, the process $(X_{-t,0}(y/v_t(\lambda)), y \geq 0)$ is a subordinator, the function κ_λ is therefore the Laplace exponent of a certain subordinator $(W^\lambda(y), y \geq 0)$. We show that there is no drift in the subordinator. Recall $\Psi'(\infty) := \lim_{u\to\infty}\frac{\Psi(u)}{u} \in (0,\infty]$. Since the case of linear branching mechanism is discarded, i.e $\Psi(q) \not\equiv bq$, one has by convexity,

$\Psi'(\infty) > \Psi'(0+)$. Choose $\delta \in (\Psi'(0+), \Psi'(\infty))$. There exists λ_0 such that for all $u \geq \lambda_0$, $\frac{\Psi(u)}{u} \geq \delta$ and thus $\frac{1}{\Psi(u)} \leq \frac{1}{\delta u}$. Therefore

$$\lim_{\theta\to\infty} \int_{\lambda_0}^{\theta} \left(\frac{1}{\Psi'(0+)u} - \frac{1}{\Psi(u)} \right) \mathrm{d}u \geq \int_{\lambda_0}^{\infty} \left(\frac{1}{\Psi'(0+)} - \frac{1}{\delta} \right) \frac{\mathrm{d}u}{u} = \infty.$$

One deduces that

$$\frac{\kappa_\lambda(\theta)}{\theta} = \frac{1}{\lambda} \exp\left(-\Psi'(0+) \int_{\lambda}^{\theta} \left(\frac{1}{\Psi'(0+)u} - \frac{1}{\Psi(u)} \right) \mathrm{d}u \right) \underset{\theta\to\infty}{\longrightarrow} 0,$$

which entails that there is no drift. Letting θ to ∞ in $\kappa_\lambda(\theta)$, we see that

$$\lim_{\theta\to\infty} \kappa_\lambda(\theta) = \nu_\lambda((0,\infty)) = e^{\Psi'(0+)\int_\lambda^\infty \frac{\mathrm{d}u}{\Psi(u)}},$$

which is finite if and only if $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} < \infty$. Therefore the Lévy measure ν_λ is finite if and only if $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} < \infty$. □

Lemma 4.14. *The process $(\hat{W}^\lambda(x), x \geq 0)$ has the same finite-dimensional law as*

$$((W^\lambda)^{-1}(x), x \geq 0),$$

where W^λ is a càdlàg subordinator with Laplace exponent κ_λ and

$$(W^\lambda)^{-1}(x) := \inf\{y \geq 0 : W^\lambda(y) > x\}.$$

Moreover, if $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} = \infty$, then the process $\hat{W}^\lambda$ has continuous paths almost surely.

Proof. By independence and stationarity of the increments of $(X_{-t,0}(y/v_t(\lambda)), y \geq 0)$, the convergence in law of the one-dimensional marginal $(X_{-t,0}(y/v_t(\lambda)), t \geq 0)$ as t goes to ∞ towards $W^\lambda(y)$, established in (4.34), entails the convergence of the finite-dimensional marginals. Since there is no drift part in the Laplace exponent κ_λ, the range of the subordinator W^λ contains almost surely no fixed point, see [Ber99, Proposition 1.9-(i)]. Hence for any $x \in (0,\infty)$ and $y \in (0,\infty)$, $\mathbb{P}(W^\lambda(y) = x) = 0$. The weak convergence of the finite-dimensional marginals of $(X_{-t,0}(y/v_t(\lambda), y \geq 0)$ entails thus that for any $0 < y_1 < \ldots < y_n$ and $0 < x_1 < \ldots < x_n$

$$\begin{aligned}&\lim_{t\to\infty} \mathbb{P}\big(X_{-t,0}(y_1/v_t(\lambda)) \geq x_1,\ X_{-t,0}(y_2/v_t(\lambda)) \geq x_2, \ldots, X_{-t,0}(y_n/v_t(\lambda)) \geq x_n\big)\\ &= \mathbb{P}\big(W^\lambda(y_1) \geq x_1,\ W^\lambda(y_2) \geq x_2,\ \ldots, W^\lambda(y_n) \geq x_n\big).\end{aligned}$$

By applying Lemma 4.10, the definition of the flow $(\hat{X}_t(x), x \geq 0)$ and the duality relation (2.12), one gets the following identities

$$\begin{aligned}(4.35)\quad &\mathbb{P}\left(\hat{W}^\lambda(x_1) \leq y_1,\ \hat{W}^\lambda(x_2) \leq y_1,\ \ldots,\ \hat{W}^\lambda(x_n) \leq y_n\right)\\ &= \lim_{t\to\infty} \mathbb{P}\left(v_t(\lambda)\hat{X}_t(x_1) \leq y_1, v_t(\lambda)\hat{X}_t(x_2) \leq y_2,\ \ldots, v_t(\lambda)\hat{X}_t(x_n) \leq y_n\right)\\ &= \lim_{t\to\infty} \mathbb{P}\big(X_{-t,0}(y_1/v_t(\lambda)) \geq x_1,\ X_{-t,0}(y_2/v_t(\lambda)) \geq x_2, \ldots, X_{-t,0}(y_n/v_t(\lambda)) \geq x_n\big)\\ &= \mathbb{P}\big(W^\lambda(y_1) \geq x_1,\ W^\lambda(y_2) \geq x_2,\ \ldots, W^\lambda(y_n) \geq x_n\big)\\ &= \mathbb{P}\left((W^\lambda)^{-1}(x_1) \leq y_1,\ \ (W^\lambda)^{-1}(x_2) \leq y_2,\ \ldots,\ (W^\lambda)^{-1}(x_n) \leq y_n\right).\end{aligned}$$

The processes $(W^\lambda)^{-1}$ and $\hat{W}^\lambda$ have therefore the same law. The fact that when $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} = \infty$, there are no jumps, i.e $J^\lambda = \emptyset$ a.s., comes from the fact that the process $(W^\lambda)^{-1}$ is the inverse of a subordinator with an infinite Lévy measure. Since there is no drift in W^λ, the sample paths of $\hat{W}^\lambda$ are pure singular continuous functions. $\square$

Proof of Theorem 3.1. The main theorem is obtained by combining Lemma 4.11, Lemma 4.14 and Lemma 4.13. $\square$

Remark 4.15. When $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} = \infty$, for any time $t>0$, the subordinator $(X_{-t,0}(x), x \geq 0)$ has an infinite Lévy measure, the process $(\hat{X}_t(x), x \geq 0)$ is therefore continuous increasing. Since its limit $(\hat{W}^\lambda(x), x \geq 0)$ is continuous, Dini's theorems ensure that the almost sure convergence (4.31) holds true locally uniformly.

Proof of Corollary 3.3. Recall the statement of Corollary 3.3. For any $\lambda > 0$, by (2.7), one has for any time t,

$$\int_{v_t(\lambda)}^{\lambda} \left(\frac{1}{\Psi'(0+)u} - \frac{1}{\Psi(u)} \right) \mathrm{d}u = \frac{1}{\Psi'(0+)} \log \left(\frac{\lambda}{v_t(\lambda)} \right) - t = \frac{1}{\Psi'(0+)} \log \left(\frac{\lambda}{v_t(\lambda)} e^{-\Psi'(0+)t} \right).$$

Recall that $v_t(\lambda) \underset{t\to\infty}{\longrightarrow} 0$. Therefore, the asymptotics $v_t(\lambda) \underset{t\to\infty}{\sim} c_\lambda e^{-\Psi'(0+)t}$ holds for some constant $c_\lambda > 0$ if and only if

$$\int_0^\lambda \left(\frac{1}{\Psi'(0+)u} - \frac{1}{\Psi(u)} \right) \mathrm{d}u < \infty. \tag{4.36}$$

One has, when (4.36) holds, $c_\lambda = \lambda e^{-\Psi'(0+) \int_0^\lambda \left(\frac{1}{\Psi'(0+)u} - \frac{1}{\Psi(u)} \right) \mathrm{d}u}$. Since $\Psi(u) \underset{u\to 0}{\sim} \Psi'(0+)u$, the convergence (4.36) is equivalent to $\int_0^\lambda \left(\frac{\Psi(u)-\gamma u}{u^2} \right) \mathrm{d}u < \infty$, where we recall that $\gamma = \Psi'(0+)$ is the linear drift in Ψ, see (2.5). Simple calculations from the Lévy-Khintchine form (2.5) ensure that the latter integral converges if and only if $\int^\infty u \log u \pi(\mathrm{d}u) < \infty$. We refer for instance to the calculations around Proposition 3.14 in [Li11]. By Theorem 3.1, almost surely for all $x \notin J^\lambda$,

$$e^{-\Psi'(0+)t} \hat{X}_t(x) \underset{t\to\infty}{\longrightarrow} \frac{1}{c_\lambda} \hat{W}^\lambda(x).$$

The process $(\frac{1}{c_\lambda} \hat{W}^\lambda(x), x \geq 0)$ is the inverse of the subordinator $(W^\lambda(c_\lambda x), x \geq 0)$, whose Laplace exponent is $\theta \mapsto c_\lambda \kappa_\lambda(\theta)$. Recall κ_λ, one easily checks that

$$c_\lambda \kappa_\lambda(\theta) = \lambda e^{-\Psi'(0+)\left[\int_0^\lambda \left(\frac{1}{\Psi'(0+)u} - \frac{1}{\Psi(u)}\right)\mathrm{d}u - \int_\lambda^\theta \frac{\mathrm{d}u}{\Psi(u)}\right]} = \theta e^{-\Psi'(0+) \int_0^\theta \left(\frac{1}{\Psi'(0+)u} - \frac{1}{\Psi(u)}\right)\mathrm{d}u}.$$

$\square$

4.3. **Proof of Theorem 3.7.** We establish now that the random partition $\mathscr{A}$ matches with the ancestral partition. The proof relies on discretizations of the current population along the sequence of jumps times of a Poisson process, see [FMM19, Section 5.2]. Consider a Poisson point process $\mathcal{P}$ on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}_+$. Let $\mu > 0$ and $(J_j^\mu, j \geq 1)$ be the sequence of atoms (i.e. of jump times) of the homogeneous Poisson process $(\mathcal{P}([0,\mu] \times [0,x]), x \geq 0)$. Note that all jump times of $(\mathcal{P}([0,\mu] \times [0,x]), x \geq 0)$ are jump times of $(\mathcal{P}([0,\mu'] \times [0,x]), x \geq 0)$ when $\mu' > \mu$.

Therefore the sequence $(J_j^{\mu'}, j \geq 1)$ is thinner than $(J_j^{\mu}, j \geq 1)$. Moreover the set of all jump times of $\mathcal{P}$, $\mathcal{M} := \{J_i^{\mu}, i \geq 1, \mu > 0\}$, is almost surely an everywhere dense subset of $[0, \infty)$. One samples now individuals in the current population along the sequence $(J_i^{\mu}, i \geq 1)$ for a fixed μ. Let $i \neq j \in \mathbb{N}$. We first establish that the individuals J_i^{μ} and J_j^{μ} have the same ancestors if and only if $J_i^{\mu} \overset{\mathscr{A}}{\sim} J_j^{\mu}$ i.e. $\hat{W}^{\lambda}(J_i^{\mu}) = \hat{W}^{\lambda}(J_j^{\mu})$. The first implication is obvious since by definition of $\hat{W}^{\lambda}$, if $\hat{X}_t(J_i^{\mu}) = \hat{X}_t(J_j^{\mu})$ for some $t \geq 0$, then $\hat{W}^{\lambda}(J_i^{\mu}) = \hat{W}^{\lambda}(J_j^{\mu})$. Denote by $(C^{\mu}(t), t \geq 0)$ the process defined as follows

$$i \overset{C^{\mu}(t)}{\sim} j \text{ if and only if } \hat{X}_t(J_i^{\mu}) = \hat{X}_t(J_j^{\mu}) \text{ for any } i, j \in \mathbb{N}^{\star}.$$

According to [FMM19, Proposition 4.18] and its proof, the process $(C^{\mu}(t), t \geq 0)$, called consecutive coalescent in [FMM19], admits an almost-sure limit $C^{\mu}(\infty)$ and for any fixed integers i and j, $i \overset{C^{\mu}(\infty)}{\sim} j$ if and only if there exists a time $t_{i,j} > 0$ such that for all $t \geq t_{i,j}$, $i \overset{C^{\mu}(t)}{\sim} j$. To establish the other implication, we will show that $C^{\mu}(\infty) = \mathscr{A}^{\mu}$ where $\mathscr{A}^{\mu}$ is defined as follows:

$$i \overset{\mathscr{A}^{\mu}}{\sim} j \text{ if and only if } \hat{W}^{\lambda}(J_i^{\mu}) = \hat{W}^{\lambda}(J_j^{\mu}).$$

Clearly all blocks of $C^{\mu}(\infty)$ are sub-blocks of $\mathscr{A}^{\mu}$. It is thus sufficient to show that the blocks of $\mathscr{A}^{\mu}$ have lengths of the same law as those of $C^{\mu}(\infty)$. Recall κ_{λ}. By [FMM19, Lemma 5.8], blocks sizes of $\mathscr{A}^{\mu}$ are i.i.d. with generating function given by,

$$\mathbb{E}[z^{\#\mathscr{A}_1^{\mu}}] = 1 - \frac{\kappa_{\lambda}(\mu(1-z))}{\kappa_{\lambda}(\mu)}, \text{ for all } z \in (0,1).$$

Simple calculations from the expression of κ_{λ}, see (3.14), entail that $\mathbb{E}[z^{\#\mathscr{A}_1^{\mu}}] = 1 - e^{-\Psi'(0+)\int_{\mu(1-z)}^{\mu} \frac{\mathrm{d}u}{\Psi(u)}}$. Applying [FMM19, Proposition 5.18], we see that $\mathbb{E}[z^{\#\mathscr{A}_1^{\mu}}] = \mathbb{E}[z^{\#C_1^{\mu}(\infty)}]$ for any $z \in (0,1)$.

It remains to deduce that $\mathscr{A}$ is indeed the ancestral partition of the whole population. If $x \overset{\mathscr{A}}{\sim} y$ and $x \neq y$, then there exists $u > 0$, such that $x, y \in \big(W^{\lambda}(u-), W^{\lambda}(u)\big)$. By density of the set $\mathcal{M}$, one can find for some large enough μ, atoms $J_x^{\mu}, J_y^{\mu} \in \mathcal{M}$ such that

$$W^{\lambda}(u-) < J_x^{\mu} < x \text{ and } W^{\lambda}(u) > J_y^{\mu} > y.$$

By construction, the individuals J_x^{μ} and J_y^{μ} belongs to the same interval of $\mathscr{A}$. We have seen above that it entails that almost surely for large enough $t \geq 0$, $\hat{X}_t(J_x^{\mu}) = \hat{X}_t(J_y^{\mu})$. Hence all individuals in the interval (J_x^{μ}, J_y^{μ}) have a common ancestor, including x and y. This achieves the proof. $\square$

Proof of Proposition 3.8. Recall the statement of Proposition 3.8, where $\mathscr{S}$ denotes the support of the random measure $\mathrm{d}\hat{W}^{\lambda}$. By Lemma 4.12, $\hat{W}^{\lambda} = c_{\lambda,\lambda'}\hat{W}^{\lambda'}$ almost surely, this entails that $\mathscr{S}$ does not depend on the parameter λ. Moreover, we see by Lemma 4.14 that $\mathscr{S}$ is the range of a subordinator W^{λ} with Laplace exponent κ_{λ}. Having the Laplace exponent of W^{λ} at hand, one can directly apply known results on the geometry of the range of a subordinator. By [Ber99, Corollary 5.3], for all $x > 0$, almost surely

$\dim_H(\mathscr{S}\cap[0,x]) = \underline{\mathrm{ind}}(\kappa_\lambda)$ with $\underline{\mathrm{ind}}(\kappa_\lambda) := \liminf_{q\to\infty} \frac{\log \kappa_\lambda(q)}{\log q}$. Recall κ_λ, one gets

$$\underline{\mathrm{ind}}(\kappa_\lambda) = \liminf_{q\to\infty} \frac{\Psi'(0+)}{\log q}\int_\lambda^q \frac{\mathrm{d}u}{\Psi(u)}.$$

If $\Psi'(\infty) := \lim_{u\to\infty} \frac{\Psi(u)}{u} < \infty$, then we see that $\int_\lambda^q \frac{\mathrm{d}u}{\Psi(u)} \underset{q\to\infty}{\sim} \frac{1}{\Psi'(\infty)}\log q$ and the result is established. If now $\Psi'(\infty)=\infty$, then for any large $D>0$, for large q,

$$\int_\lambda^q \frac{\mathrm{d}u}{\Psi(u)} \leq C_\lambda + \frac{\log q}{D},$$

for some constant C_λ. We see therefore that $\underline{\mathrm{ind}}(\kappa_\lambda) \leq \frac{\Psi'(0+)}{D}$. Since D is arbitrarily large, one can conclude that $\dim_H(\mathscr{S}\cap[0,x]) = \frac{\Psi'(0+)}{\Psi'(\infty)} = 0$ a.s. □

We now establish Proposition 3.11 and look for the density of $\hat{W}^\lambda(x)$ for fixed x.

Proof of Proposition 3.11. Let $\lambda>0$ and $\theta>0$. Recall κ_λ and denote by $\mathbb{e}_\theta$, $\mathbb{e}_q$ two independent exponential random variables with parameter θ and q respectively. By Lemma 4.14, we have that $\mathbb{P}(W^\lambda(\mathbb{e}_\theta)\geq \mathbb{e}_q) = \mathbb{P}(\mathbb{e}_\theta \geq \hat{W}^\lambda(\mathbb{e}_q))$, or equivalently

$$\mathbb{E}[e^{-\theta \hat{W}^\lambda(\mathbb{e}_q)}] = 1-\mathbb{E}[e^{-qW^\lambda(\mathbb{e}_\theta)}].$$

We deduce that

$$\begin{aligned}\int_0^\infty \mathbb{E}[e^{-\theta\hat{W}^\lambda(x)}]e^{-qx}\mathrm{d}x &= \frac{1}{q}\left(1-\mathbb{E}[e^{-\kappa_\lambda(q)\mathbb{e}_\theta}]\right) = \frac{1}{q}\left(1-\frac{\theta}{\kappa_\lambda(q)+\theta}\right) = \frac{1}{\theta}\frac{\kappa_\lambda(q)}{q}\frac{\theta}{\kappa_\lambda(q)+\theta}\\ &= \frac{1}{\theta}\frac{\kappa_\lambda(q)}{q}\mathbb{E}[e^{-\kappa_\lambda(q)\mathbb{e}_\theta}] = \frac{1}{\theta}\int_0^\infty e^{-qu}\bar{\nu}_\lambda(u)\mathrm{d}u\int_0^\infty e^{-qz}\mathbb{P}(W^\lambda(\mathbb{e}_\theta)\in\mathrm{d}z),\end{aligned}$$

where we have used that

$$\mathbb{E}[e^{-qW^\lambda(\mathbb{e}_\theta)}] = \mathbb{E}[e^{-\kappa_\lambda(q)\mathbb{e}_\theta}] \text{ and } \frac{\kappa_\lambda(q)}{q} = \int_0^\infty e^{-qu}\bar{\nu}_\lambda(u)\mathrm{d}u.$$

By the change of variable $x=u+z$, we obtain

$$\int_0^\infty \mathbb{E}[e^{-\theta\hat{W}^\lambda(x)}]e^{-qx}\mathrm{d}x = \frac{1}{\theta}\int_0^\infty e^{-qx}\int_0^\infty \bar{\nu}_\lambda(x-z)\mathbb{P}(W^\lambda(\mathbb{e}_\lambda)\in\mathrm{d}z),$$

and deduce that

$$\begin{aligned}\mathbb{E}[e^{-\theta\hat{W}^\lambda(x)}] &= \frac{1}{\theta}\int_0^x \bar{\nu}_\lambda(x-z)\mathbb{P}(W^\lambda(\mathbb{e}_\theta)\in\mathrm{d}z)\\ &= \int_0^\infty e^{-\theta u}\int_0^x \bar{\nu}_\lambda(x-z)\mathbb{P}(W^\lambda(u)\in\mathrm{d}z)\mathrm{d}u.\end{aligned}$$

Thus, the density of $\hat{W}^\lambda(x)$ is $g_x^\lambda(u) := \int_0^x \bar{\nu}_\lambda(x-z)\mathbb{P}(W^\lambda(u)\in\mathrm{d}z)$. In the case $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} < \infty$, $(W^\infty(u), u\geq 0)$ is a compound Poisson process with intensity ν_∞, and since $\bar{\nu}_\infty(0)=1$, the formula (3.17) can be plainly checked. □

Acknowledgements: C.F's research is partially supported by LABEX MME-DII (ANR11-LBX-0023-01).

References

[AS96] Soren Asmussen and Karl Sigman, *Monotone stochastic recursions and their duals*, Probab. Engrg. Inform. Sci. 10 (1996), no. 1, 1-20. 2

[Bar75] Andrew D. Barbour, *The asymptotic behaviour of birth and death and some related processes*, Advances in Appl. Probability **7** (1975), 28–43. 14

[Ber99] Jean Bertoin, *Subordinators: Examples and Applications*, pp. 1–91, Springer Berlin Heidelberg, Berlin, Heidelberg, 1999. 18, 20

[BFM08] Jean Bertoin, Joaquin Fontbona, and Servet Martínez, *On prolific individuals in a supercritical continuous-state branching process*, J. Appl. Probab. **45** (2008), no. 3, 714–726. 2

[BLG00] Jean Bertoin and Jean-François Le Gall, *The Bolthausen-Sznitman coalescent and the genealogy of continuous-state branching processes*, Probab. Theory Related Fields **117** (2000), no. 2, 249–266. 1, 4

[Bin76] N. H. Bingham, *Continuous branching processes and spectral positivity*, Stochastic Processes Appl. **4** (1976), no. 3, 217–242. 1, 5

[BGT87] N. H. Bingham, C. M. Goldie, and Jef L. Teugels, *Regular variation*, Encyclopedia of mathematics and its applications, Cambridge University Press, Cambridge, Cambridgeshire, New York, 1987, Includes indexes. 14

[Chu68] Kai Lai Chung, *A Course in Probability Theory*, Harcourt, Brace & World, Inc., New York, 1968. 14

[CR84] J. Theodore Cox and Uwe Rösler, *A duality relation for entrance and exit laws for Markov processes*, Stochastic Process. Appl. **16** (1984), no. 2, 141–156.

[CS85] Peter Clifford and Aidan Sudbury, *A sample path proof of the duality for stochastically monotone Markov processes*, Ann. Probab. **13** (1985), no. 2, 558–565. 12

[Cur42] J. H. Curtiss, *A note on the theory of moment generating functions*, Ann. Math. Statistics **13** (1942), 430–433. 2

[DL14] Thomas Duquesne and Cyril Labbé, *On the Eve property for CSBP*, Electron. J. Probab. **19** (2014), no. 6, 31. 14

[Fel71] William Feller, *An Introduction to Probability Theory and Its Applications. Vol. II*, Second edition, John Wiley & Sons, Inc., New York-London-Sydney, 1971. 2

[FM19] Clément Foucart and Chunhua Ma, *Continuous-state branching processes, extremal processes and super-individuals*, Ann. Inst. H. Poincaré Probab. Statist. **55** (2019), no. 2, 1061–1086. 10, 14

[FMM19] Clément Foucart, Chunhua Ma, and Bastien Mallein, *Coalescences in continuous-state branching processes*, Electron. J. Probab. **24** (2019), no. 103. 2

[Gre74] D. R. Grey, *Asymptotic behaviour of continuous time, continuous state-space branching processes*, J. Appl. Probab. **11** (1974), 669–677. 2, 5, 6, 8, 9, 10, 19, 20

[Jiř58] Miloslav Jiřina, *Stochastic branching processes with continuous state space*, Czechoslovak Math. J. **8 (83)** (1958), 292–313. 1, 7

[JL21+] Samuel G. G. Johnston and Amaury Lambert, *The coalescent structure of uniform and Poisson samples from multitype branching processes*, ArXiv eprint 1912.00198, (2021). 3

[Lab14] Cyril Labbé, *Genealogy of flows of continuous-state branching processes via flows of partitions and the eve property*, Ann. Inst. H. Poincaré Probab. Statist. **50** (2014), no. 3, 732–769. 2

[Lag05] Andreas Nordvall Lagerås, *A renewal-process-type expression for the moments of inverse subordinators*, J. Appl. Probab. **42** (2005), no. 4, 1134–1144. 2

[Lam67] John Lamperti, *Continuous state branching processes*, Bull. Amer. Math. Soc. **73** (1967), 382–386. 10

[Lam03] Amaury Lambert, *Coalescence times for the branching process*, Adv. in Appl. Probab. **35** (2003), no. 4, 1071–1089. 3

[Lam07] Amaury Lambert, *Quasi-stationary distributions and the continuous-state branching process conditioned to be never extinct*, Electron. J. Probab. **12** (2007), no. 14, 420–446. 2

[Li00] Zeng-Hu Li, *Asymptotic behaviour of continuous time and state branching processes*, J. Austral. Math. Soc. Ser. A **68** (2000), no. 1, 68–84. 4

[Li11] Zenghu Li, *Measure-valued branching Markov processes*, Probability and its Applications (New York), Springer, Heidelberg, 2011. 4

[LP13] Amaury Lambert and Lea Popovic, *The coalescent point process of branching trees*, Ann. Appl. Probab. **23** (2013), no. 1, 99–144. 3, 7, 19

[LPLG08] Yangrong Li, Anthony G. Pakes, Jia Li, and Anhui Gu, *The limit behavior of dual Markov branching processes*, J. Appl. Probab. **45** (2008), no. 1, 176–189. 2

[Pak17] Anthony G. Pakes, *Convergence rates and limit theorems for the dual Markov branching process*, J. Probab. Stat. (2017), 13. 2

[Sie76] D. Siegmund, *The equivalence of absorbing and reflecting barrier problems for stochastically monotone Markov processes*, Ann. Probab. **4** (1976), no. 6, 914–924. 2, 14

[Sil68] M. L. Silverstein, *A new approach to local times*, J. Math. Mech. **17** (1967/1968), 1023–1054. 2, 6, 11

[VT10] Mark Veillette and Murad S. Taqqu, *Using differential equations to obtain joint moments of first-passage times of increasing Lévy processes*, Statistics and Probability Letters **80** (2010), no. 7, 697 – 705. 3

[Wid41] D. V. Widder, *Laplace Transform*, Mathematics Series, Princeton Univ. Press, 1941. 10

12