Université Sorbonne Paris Nord

École doctorale Galilée

HABILITATION À DIRIGER DES RECHERCHES

Spécialité : Mathématiques

présentée par

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Quelques contributions à l'étude des processus de branchement et coalescence par dualité

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Remerciements

Je tiens tout d'abord à exprimer ma profonde gratitude à Bénédicte Haas, Andreas Kyprianou et Viet Chi Tran qui ont accepté d'être rapporteurs. Merci d'avoir consacré du temps à la lecture de ce manuscrit. Vos remarques et conseils avisés ont été d'une grande aide, avant, pendant et après la rédaction. Je remercie également chaleureusement Sylvie Méléard, Jean-François Delmas, Amaury Lambert et Thomas Duquesne pour leur présence dans le jury.

Ce manuscrit repose sur des travaux que j'ai menés depuis mon arrivée au LAGA en 2013. Le laboratoire et l'équipe de probabilité et statistique m'ont offert un cadre scientifique à la fois stimulant et bienveillant. Je remercie l'ensemble des collègues du laboratoire pour leur sympathie, leur générosité et la richesse de nos échanges. Merci également à Yolande et Leïla pour leur gentillesse et leur aide quotidienne dans la gestion des missions. La liste serait bien évidemment trop longue pour citer tous les collègues, je tiens néanmoins à remercier tout particulièrement Julien Barral, Bénédicte Haas, Yueyun Hu, Bastien Mallein, Emmanuel Roy, Laurent Tournier, Mohamed Ben-Alaya et Ahmed Kebaier.

Une grande partie de mes recherches a été réalisée en collaboration, et je souhaite ici remercier mes co-auteurs pour tout ce qu'ils m'ont appris : Xan Duhalde, Olivier Hénard, Pei-Sen Li, Bo Li, Chunhua Ma, Bastien Mallein, Martin Möhle, Victor Rivero, Gerónimo Uribe Bravo, Matija Vidmar, Linglong Yuan, Anita Winter et Xiaowen Zhou.

Mon détachement au CMAP depuis septembre 2023, au sein de l'ERC de Sylvie Méléard, a joué un rôle déterminant dans l'avancement et la finalisation de cette HDR. Sylvie, je te remercie infiniment pour cette opportunité. Les discussions au sein de l'ERC SINGER, notamment avec toi, Anita, Sarah, Vincent, Nicolas, Chi, ont nourri ma réflexion, influencent et influenceront sans aucun doute mes futures recherches. Je tiens également à remercier chaleureusement mes collègues de l'équipe PEIPS – en particulier Carl, Cyril, Igor, Lucas, Marie, Gaël, Milica et Yoan – pour leur accueil à Polytechnique et nos discussions, qu'elles soient d'ordre mathématique ou plus personnelles. Un grand merci à mon doctorant Félix Rebotier pour la confiance qu'il m'accorde dans la direction de sa thèse.

Un grand merci à mon père, qui m'a transmis le goût des mathématiques et m'a soutenu tout au long de ce parcours.

A ma compagne Nadira et mon fils Thomas, les deux personnes qui comptent le plus pour moi, je vous adresse tout mon amour.

Résumé Abstract

Quelques contributions à l'étude des processus de branchement et de coalescence par dualité

L'objet de ce manuscrit est de présenter une synthèse de mes travaux publiés depuis 2013. On trouvera la liste complète des publications et pré-publications ci-dessous. Afin de donner un fil conducteur à ce document, je me concentrerai sur une sélection d'articles. Les publications [F4, F5, F6, F10, F11, F14, F19] et les pré-publications [F20, F21, F22] ne seront pas décrites en détail dans ce document mais mentionnées lorsque cela s'y prête. Les publications issues de ma thèse [F1-F3] ne seront pas présentées.

Les deux notions mathématiques centrales du document sont les processus de branchement markoviens et les coalescents échangeables.

Nous étudions tout d'abord la généalogie des processus de branchement en temps et espace continus (CSBPs) à l'aide des flots de subordinateurs de Bertoin et Le Gall. Nous verrons comment renormaliser presque sûrement le flot « descendant » et comment interpréter le processus limite du point de vue de la généalogie. Nous étudions ensuite le flot *inverse* pour décrire la généalogie ascendante de la population branchante. Nous verrons comment suivre les lignées ancestrales dans le flot à l'aide d'un certain processus de Markov à sauts négatifs. Un coalescent Markovien non-échangeable élémentaire, permettant de suivre les coalescences des lignées, sera défini.

Dans un deuxième temps, nous nous intéresserons à des généralisations des CSBPs incorporant des interactions et à la classification de leurs comportements en temps long et aux frontières. Le cas des CSBPs logistiques (c'est-à-dire avec compétition quadratique) est étudié en détail et un phénomène de réflexion à l'infini est mis en évidence. Nous introduisons ensuite une classe plus générale d'interactions appelées collisions et étudions, entre autres choses, leurs premiers temps de passage sous un niveau fixé.

Parallèlement, les processus de fragmentation-coalescence échangeables (EFC) et les processus comptant leur nombre de blocs sont étudiés. Nous trouvons des conditions suffisantes pour que le processus descende de l'infini, reste infini ou explose. Nous verrons enfin les processus de Wright-Fisher généralisés avec sélection et donnerons des conditions suffisantes pour voir apparaître différents comportements aux bords.

Dés que nous étudierons les lignées ancestrales des CSBPs, ou que nous ajouterons des interactions ou des coalescences, la propriété de branchement sera perdue. Celle-ci sera en quelque sorte remplacée par une notion de dualité entre processus. Nous verrons comment plusieurs relations de dualités permettent d'étudier de nombreux problèmes pour chacun des processus que nous avons mentionnés.

Mots-clefs

Processus de branchement, Processus de coalescence, Subordinateur, Processus extrémal, Premier temps de passage, Dualité, Généalogie.

Some contributions to the study of branching and coalescence processes by duality

The purpose of this manuscript is to present a synthesis of my works published since 2013. The complete list of publications and pre-publications can be found below. To give a common thread to this document, I will focus on a selection of articles. The publications [F4, F5, F6, F10, F11, F14, F19] and preprints [F20, F21, F22] will not be described in detail in this document but mentioned when it sounds appropriate. The publications resulting from my thesis [F1-F3] are not discussed.

The two central mathematical notions of the document are Markov branching processes and exchangeable coalescents.

We first study the genealogy of branching processes in continuous time and space (CSBPs) using the flow of subordinators of Bertoin and Le Gall. We will see how to renormalize the forward "flow" and how to interpret the limit process from the point of view of genealogy. We then study the *inverse* flow in order to describe the genealogy of the branching population backwards in time. We will see how to follow ancestral lines in the flow with a certain negative jump Markov process. An elementary non-exchangeable Markovian coalescent, making it possible to follow coalescences between lineages, is defined.

Secondly, we are interested in generalizations of CSBPs incorporating interactions and the classification of their behaviors in long time and at boundaries. The case of logistic CSBPs (i.e. with quadratic competition) is studied in detail and a phenomenon of reflection at infinity is highlighted. We then introduce a more general class of interactions called collisions and study, among other things, their first times of passage below a fixed level.

Next, exchangeable fragmentation-coalescence (EFC) processes and the process counting their number of blocks are studied. We find sufficient conditions for the process to come down from infinity or to stay infinite. In the same fashion, we investigate also its possible explosion. We will finally study the Wright-Fisher processes generalized with selection and will give sufficient conditions to see different behaviors at the boundaries.

As soon as we will be studying the ancestral lineages of a CSBP backwards in time, or add interactions or coalescences into the dynamics, the branching property will be lost. The latter, in some way, will be replaced by a notion of duality between processes. We will see how several duality relations allow us to study each of the processes we have mentioned.

Keywords

Branching process; Coalescence process; Subordinator; Extremal process; First passage time; Duality; Genealogy.

Liste de mes publications et prépublications par ordre chronologique de pré-publication.

Publications

Ces publications sont disponibles dans leurs dernières versions avant publication sur ma page web, ainsi que sur ArXiv et Hal.

[F1] C. Foucart. Distinguished exchangeable coalescents and generalized Fleming-Viot processes with immigration. *Advances in Applied Probability*, (2011).

[F2] C. Foucart. Generalized Fleming-Viot processes with immigration via stochastic flows of partitions. *ALEA, Lat. Am. J. Probab. Math. Stat.* (2012).

[F3] C. Foucart and O. Hénard. Stable continuous-state branching processes with immigration and Beta-Fleming-Viot processes with immigration, *Electronic Journal of Probability*, (2013).

[F4] C. Foucart. The impact of selection in the Λ-Wright-Fisher model, *Electronic Communication in Probability* (2013) (+ Erratum).

[F5] C. Foucart and G. Uribe Bravo, Local extinction in continuous-state branching process with immigration *Bernoulli*, (2014).

[F6] X. Duhalde, C. Foucart and C. Ma On the hitting time of continuous-state branching processes with immigration, *Stochastic Processes and their Application* (2014).

[F7] C. Foucart and C. Ma. Continuous-state branching processes, extremal processes and super-individuals, *Ann. Inst. Henri Poincaré Probab. Stat*, (2019).

[F8] C. Foucart. Continuous-state branching processes with competition : duality and reflection at infinity, *Electronic journal of Probability*, (2019).

[F9] C. Foucart, C. Ma and B. Mallein. Coalescences in Continuous-State Branching Processes. *Electronic journal of Probability* (2019).

[F10] C. Foucart, P-S. Li and X. Zhou. On the entrance at infinity of Feller processes with no negative jumps, *Statistics and Probability Letters* (2020).

[F11] C. Foucart, P-S. Li and X. Zhou. Time-changed spectrally positive Lévy processes started at infinity. *Bernoulli* (2021).

[F12] C. Foucart. A phase transition in the coming down from infinity of simple exchangeable fragmentation-coagulation processes. *Annals of Applied Probability* (2022).

[F13] C. Foucart and X. Zhou. On the explosion of the number of fragments in the simple exchangeable fragmentation-coalescence processes. *Ann. Inst. Henri Poincaré Probab. Stat.* (2022).

[F14] C. Foucart, C. Ma and L. Yuan. Limit theorems for continuous-state branching processes with immigration. *Advances in Applied Probability* (2022).

[F15] C. Foucart and M. Möhle. Asymptotic behaviour of ancestral lineages in subcritical continuous-state branching populations. *Stochastic Processes and Their Applications.* (2022).

[F16] C. Foucart and X. Zhou. On the boundary classification of Λ-Wright-Fisher processes with frequency-dependent selection. *Annales Henri Lebesgue*, (2023).

[F17] C. Foucart and M. Vidmar. Continuous-state branching processes with collisions : first passage times and duality, *Stochastic Processes and their Applications* (2024).

[F18] C. Foucart and L. Yuan. Extremal shot noise processes and random cutout sets. Accepted in *Bernoulli.*

[F19] C. Foucart and L. Yuan. Weak convergence of continuous-state branching processes with large immigration. Accepted in *Stochastic Processes and Their Applications.*

Prépublications

Les travaux ci -dessous ont été soumis pour publication et sont disponibles en ligne :

[F20] C. Foucart. Local explosions and extinction in continuous-state branching processes with logistic competition.

[F21] C. Foucart, B. Li and X. Zhou. Behaviors near explosion of nonlinear CSBPs with regularly varying mechanisms.

[F22] C. Foucart, Conditioning the logistic continuous-state branching process on non-extinction via total progeny, with Victor Rivero and Anita Winter.

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Introduction

Les processus de branchement en temps et espace continus, on utilisera l'acronyme anglais CSBPs, sont des processus de Markov réels positifs représentant l'évolution de la taille d'une population aléatoire continue dans laquelle les individus se reproduisent indépendamment et de la même façon. Ils généralisent la diffusion branchante de Feller (1950), voir [59], et jouent depuis lors un rôle important dans de très nombreux modèles aléatoires (modèles de population, modèles d'énergie, marches aléatoires branchantes, super-processus, arbres aléatoires, coalescents échangeables, cartes aléatoires,...).

Les CSBPs et leurs propriétés trajectorielles ont suscité beaucoup d'attention depuis la fin des années 1970 et l'article fondateur de Grey [68]. Ces propriétés présentent de nombreuses similitudes avec celles des processus de Bienaymé-Galton-Watson. On retrouve par exemple la dichotomie extinction/non-extinction ainsi que les critères $L \log L$ de croissance exponentielle connus dans le cadre du temps et de l'espace discret, voir Kesten-Stigum [85], Seneta [133] et Heyde [76]. Il convient toutefois de noter que dans le cadre des CSBPs, le temps continu permet d'exploser, au sens où le processus atteint l'infini en temps fini par accumulation de grands sauts. L'espace continu quant à lui rend possible une convergence du processus vers zéro, en lieu et place d'une absorption en temps fini, on parle parfois d'extinction asymptotique, la frontière zéro dans ce cas est un point attractif inaccessible.

Autour des années 2000, des travaux profonds ont été menés pour comprendre la généalogie de la population associée aux CSBPs. Cela a conduit à de nombreux résultats sur les arbres aléatoires de Lévy [47] et sur certains flots de subordinateurs emboîtés [21]. Ces deux objets aléatoires permettent de définir la généalogie descendante des individus évoluant dans la population branchante continue. L'étude de la généalogie des ancêtres en remontant le temps, autrement dit de la généalogie *ascendante*, a été abordée par de nombreux auteurs sous différents angles et continue encore d'être étudiée.

Les CSBPs ont été depuis lors généralisés dans différentes directions, avec notamment des dynamiques d'immigration, des catastrophes, de l'environnement aléatoire, ou de la compétition. Les phénomènes d'immigration et d'environnement aléatoire préservent une certaine propriété de branchement. Cette dernière est par contre brisée lorsqu'un terme non-linéaire de compétition est ajouté. En 2005, dans un travail précurseur [92], Lambert a introduit un processus, notons le Z, modélisant la taille d'une population dans laquelle des morts quadratiques sont ajoutées via un terme de dérive négative « $-\frac{c}{2}Z_t^2 \mathrm{d}t$ ». Ce processus, baptisé CSBP *logistique*, peut s'interpréter comme la taille d'une population partageant des ressources limitées. Nous verrons ensuite la classe plus générale des CSBPs avec *collisions*, dans lesquels, heuristiquement, les individus « collisionnent » par paire et laissent une masse aléatoire (sous)-critique d'individus.

De façon générale, l'ajout d'une dynamique non-linéaire pose de nombreuses nouvelles questions sur le processus et son comportement aux frontières et en temps long, ainsi que sur la description de la généalogie de la population sous-jacente. Plusieurs travaux récents ont cherché à étudier cette dernière, citons par exemple les travaux de Berestycki et al. [14], Le et al. [100] et Li et al. [111] sur les arbres aléatoires « sous attaque » et les représentations de type Ray-Knight des CSBPs avec compétition.

Parallèlement à l'émergence des arbres aléatoires continus, les processus échangeables de coalescence et de fragmentation ont vu le jour ; voir à ce sujet l'ouvrage de Bertoin [18]. Il s'agit de processus Markoviens à valeurs dans les partitions d'entiers dont les classes d'équivalence (les blocs) coalescent ou se fragmentent au cours du temps, sans que la forme des blocs n'entre en compte. Le phénomène de descente de l'infini des coalescents a été très étudié. Celui-ci stipule que partant d'une partition avec une infinité de blocs (par exemple les singletons), certains processus voient leurs blocs fusionner tellement vite qu'il n'y en a plus qu'un nombre fini instantanément. Une condition nécessaire et suffisante pour la descente de l'infini des coalescents purs a été obtenue par Schweinsberg [131] lorsqu'il n'y a pas de coalescences multiples simultanées (Λ-coalescents).

Les liens entre les processus coalescents et les généalogies ascendantes de certaines populations (souvent supposées de taille finie constante) ont été beaucoup regardés. Les coalescents échangeables décrivent en fait la généalogie des processus de Wright-Fisher (ou de Fleming-Viot) généralisés, connus en génétique théorique, pour représenter la fréquence d'un allèle sans avantage sélectif (i.e. neutre) dans un modèle à deux allèles. La population ici n'est pas branchante mais évolue par *ré-échantillonnage* : à chaque instant de reproduction, un individu est tiré au hasard dans la population et donne son allèle à une fraction de la population. Birkner et al. [26] ont montré qu'après renormalisation et un changement de temps *aléatoire* approprié, les CSBPs stables ont même loi que certains processus de Wright-Fisher. La généalogie de ces derniers est décrite par la classe des Béta-coalescents. On peut ainsi relier les CSBPs stables aux coalescents. Ceci n'est cependant vrai que dans les cadres stables (et le cas « limite stable » de Neveu). D'autre part la généalogie du CSBP ainsi obtenue est décrite après renormalisation par la taille et le long d'un changement de temps aléatoire (dépendant de la taille totale de la population).

De façon assez similaire au phénomène de compétition dans le branchement, les processus de Wright-Fisher ont été généralisés afin de prendre en compte l'effet de la sélection naturelle sur un allèle délétère. On parle ainsi de processus de Wright-Fisher (WF) avec sélection. Nous renvoyons par exemple au cours de Saint-Flour d'Etheridge [52], au travail récent de Cordero et al. [41], ainsi qu'aux références qui s'y trouvent.

Dans ce bref et très général état de l'art, nous avons en fait introduit les processus de Markov étudiés dans ce document et exposer les problématiques générales qui vont nous intéresser. Nous les décrivons plus en détails ci-dessous.

Organisation du manuscrit

Le manuscrit comporte trois parties indépendantes, chacune ayant deux chapitres. Si des liens forts existent entre les chapitres, ceux-ci peuvent néanmoins être parcourus de manière autonome. Des esquisses de preuve sont données pour quelques résultats.

La partie I concerne la généalogie des CSBPs classiques ([F7],[F9] et [F15]).

Nous étudions tout d'abord dans le Chapitre 1 la population dans le sens normal du temps et trouvons une renormalisation non-linéaire presque-sûre du flot lorsque le CSBP a une moyenne (ou une variation) infinie. Le processus limite s'avère être un processus extrémal (i.e. le processus des records d'un certain processus de Poisson ponctuel) et ses temps de sauts forment une famille particulière d'individus que nous décrirons.

Nous étudions ensuite, dans le Chapitre 2, la généalogie ascendante de la population en prenant comme point de départ, le flot de subordinateurs $(X_{s,t}(z), s \leq t, z \geq 0)$ de Bertoin et Le Gall [21]. Celui-ci permet de considérer une population branchante stationnaire de taille *fixe infinie* indexée par $\mathbb{R}$ et de suivre les lignées ancestrales en renversant le temps et en inversant les subordinateurs. Nous étudions les propriétés fondamentales du flot, noté $(\widehat{X}_{s,t}(x), s < t, x \geq 0)$, ainsi obtenu. Si l'on considère un individu x aujourd'hui, le processus de Markov issu de x, $(\widehat{X}_t(x), t \geq 0) := (\widehat{X}_{0,t}(x), t \geq 0)$ suit les ancêtres de x en remontant dans le passé. Nous appellerons $\widehat{X}$ le processus de la lignée ancestrale (ALP). Nous encodons ensuite les coalescences des lignées ancestrales à l'aide d'un coalescent markovien non-échangeable élémentaire, puis nous étudions la partition ancestrale, c'est-à-dire les familles d'individus pris à l'instant 0 ayant différents ancêtres communs.

La partie II traite des CSBPs logistiques et des collisions ([F8], [F20] et [F17]).

Dans le Chapitre 3, nous reprenons le cadre des CSBPs logistiques définis dans [92]. Nous étudions tout d'abord leur explosion et trouvons une condition nécessaire et suffisante sur le mécanisme de branchement pour que celle-ci ait lieu. L'existence de lois de reproduction pouvant battre la compétition n'était pas garantie ! Dans un deuxième temps nous construisons et étudions le comportement en temps long d'une *extension* Markovienne du processus après sa première explosion. Des conditions nécessaires et suffisantes sont données pour que la frontière infinie du processus étendu se trouve être un point régulier réfléchissant et régulier pour lui-même. On identifiera en loi le temps local en l'infini du processus avec celui en zéro d'une certaine diffusion explicite et nous initierons l'étude de la mesure d'excursion en dehors de l'infini.

Dans le Chapitre 4, nous généralisons le processus logistique en ajoutant une dynamique de collisions aléatoires. L'accent ici est mis non pas sur la classification des frontières mais sur l'étude des premiers temps de passage, de la stationnarité et des autres comportements en temps long. Beaucoup d'informations sur ces processus sont disponibles !

La partie III porte sur les processus de fragmentation-coalescence et les processus de Wright-Fisher avec sélection ([F12], [F13], [F16]).

Les processus de fragmentation-coalescence échangeables (EFC) ont été définis en 2005 par Julien Berestycki [11]. Dans ces processus les deux phénomènes de fragmentation et de coagulation sont pris en compte. Nous étudierons dans le Chapitre 5, la frontière infinie du processus comptant le nombre de blocs des EFCs. L'esprit est similaire à celui du Chapitre 3, les méthodes sont par contre différentes. En supposant que le coalescent seul descend de l'infini, nous verrons en quelque sorte comment les coalescences et les fragmentations doivent se combiner, soit pour avoir descente de l'infini, soit pour empêcher le processus de sortir de l'infini.

De façon un peu symétrique, nous trouverons des conditions pour que la fragmentation fasse exploser le nombre de blocs, même lorsque la coalescence est très forte. Dans les cas où les mesures de coalescence et de fragmentation sont stables, on trouvera des conditions explicites délimitant trois régimes possibles pour le point ∞ : celui-ci peut être un point d'entrée (le processus en sort mais n'y accède pas), un point de sortie (Il y accède mais ne peut pas en ressortir) ou un point régulier (Le processus sort du point et y retourne).

Dans le Chapitre 6, non sans lien avec le précédent, nous étudions les processus de Wright-Fisher avec sélection et leur possible extension après avoir touché le point frontière 1. Les connexions mentionnées plus haut entre les coalescents échangeables et les processus de Wright-Fisher vont en quelque sorte se généraliser au cadre avec sélection et nous pourrons utiliser les résultats obtenus sur les EFCs pour étudier ces processus.

Chaque chapitre se conclut par une section de commentaires, où sont brièvement évoqués des liens avec d'autres travaux et où nous suggérons des pistes de recherche potentielles.

Les démonstrations de la plupart des résultats présentés dans ce manuscrit reposent sur le concept de *dualité*. Cette notion clé transfèrera certaines propriétés du processus qui nous intéresse (telles que l'accessibilité d'un point, la transience, la récurrence) en d'autres propriétés pour son processus dual. Des correspondances vont ainsi apparaître et permettront d'étudier des processus de Markov à sauts non triviaux (et disons pour lesquels aucune théorie générale et explicite n'est à disposition) en étudiant les processus en dualité. Dans le meilleur des cas, les processus duaux font parties d'une classe très bien comprise (les diffusions par exemple). Lorsqu'aucun des processus n'est sensiblement plus facile à étudier que son dual, tout effort d'un côté sera récompensé. Cela s'illustrera en particulier dans le dernier chapitre.

Dualité entre processus.
Etant donnés deux processus markoviens X et Y, et une fonction H, ceux-ci sont dits en H-dualité *au niveau des semi-groupes*, si pour tous x, y et $t \geq 0$,

$$\mathbb{E}_x[H(X_t, y)] = \mathbb{E}_y[H(x, Y_t)].$$

Ce concept de dualité est classique dans la théorie des super-processus ainsi que dans l'étude des systèmes de particules en interaction.

Nous rencontrerons trois fonctions H :

1. **Dualité de Laplace** : $\forall x, z \in (0, \infty)$, $H(x, z) := e^{-xz} =: e_x(z) =: e_z(x)$.
2. **Dualité de Siegmund** : $\forall x, y \in (0, \infty)$, $H(x, y) := \mathbb{1}_{\{x<y\}}$.
3. **Dualité des moments** : $\forall x \in (0, 1),\ n \in \mathbb{N}$, $H(x, n) := x^n =: f_x(n) =: f_n(x)$.

Ces trois fonctions déterminent les lois unidimensionnelles des processus. Les relations de H-dualité caractérisent ainsi les semi-groupes des processus.

Le premier exemple de relation de dualité de Laplace que nous utiliserons n'est autre que celle donnée par la propriété de branchement : si X est un CSBP, alors sa transformée de Laplace prend la forme suivante

$$\mathbb{E}_z[e^{-xX_t}] = e^{-zu_t(x)},$$

avec $(u_t(x), t \geq 0, x \in (0, \infty))$ un certain flot déterministe appelé **cumulant**. Le processus dégénéré $u := (u_t, t \geq 0)$ est le dual de Laplace de X.

Nous verrons ensuite que les processus de branchement avec compétition ou collision, notés Z dans tout le texte, vérifient (en précisant les conditions aux bords !) une dualité de Laplace avec une *diffusion* positive U :

$$\mathbb{E}_z[e^{-xZ_t}] = \mathbb{E}_x[e^{-zU_t}].$$

L'étude du processus U, de la même manière que dans le cas des processus de branchement, où l'on examine souvent le cumulant u, permettra d'aborder de nombreuses questions concernant Z. Nous verrons en particulier comment la dualité met en correspondance les différents types de frontières (entrée, sortie, régulier, naturel). Cette terminologie est ré-expliquée au fil du texte et tous les types de frontières abordés sont décrits en Annexe.

Nous rencontrerons la dualité de Siegmund lors de l'étude de la généalogie ascendante des CSBPs (Chapitre 2). Le processus de la lignée ancestrale $\widehat{X}$ est le dual de Siegmund du CSBP X :

$$\mathbb{P}_x(\widehat{X}_t < z) = \mathbb{P}_z(X_t > x).$$

Le processus $\widehat{X}$ sera en fait défini sur le même espace de probabilité que X (celui où vit le flot). On parle de dualité *trajectorielle*.

Nous verrons également qu'il peut être intéressant d'utiliser deux relations de dualité successivement. Par exemple dans le cas des CSBPs logistiques (LCSBP) et des CSBPs avec collision, on introduira le processus V en dualité de Siegmund avec U :

$$Z \overset{\text{Laplace dual}}{\longleftrightarrow} U \overset{\text{Siegmund dual}}{\longleftrightarrow} V.$$

On appellera V le processus *bidual* de Z.

Une étude indépendante de la dualité de Siegmund pour les diffusions sera faite dans le Chapitre 3.4. Celle-ci permettra d'identifier explicitement le processus V. Celui-ci s'avérera être beaucoup plus simple à étudier que Z.

La dualité des moments, quant à elle, sera utile dans l'étude des processus de Wright-Fisher à la fin du document. Nous noterons F le processus de Wright-Fisher avec sélection et N (ou $\#\Pi$) le nombre de blocs d'un EFC :

$$\mathbb{E}_x(F_t^n) = \mathbb{E}_n(x^{N_t}).$$

Une fois collectées les informations concernant la frontière infinie de N, nous les transférerons pour la frontière 1 de F (qui correspond à la fixation de l'allèle désavantagé). Nous construirons une extension du processus F avec une frontière 1 non-absorbante (régulière ou entrée) qui vérifiera une dualité des moments avec le processus N *arrêté* à son premier temps d'explosion.

L'explication des différents rôles joués par la dualité que nous venons de donner mène naturellement aux deux problèmes suivants, non étudiés dans ce document :

— **Généalogie et dualité trajectorielle**. Dans le cadre des processus de Wright-Fisher la dualité des moments peut se comprendre à l'aide de la notion d'*Ancestral Selection Graph* généralisée et du concept de « sampling duality », voir par exemple l'article de Gonzalez-Casanova et Spanó [36]. La question naturelle de comprendre généalogiquement les relations de dualités de Laplace et de Siegmund pour les processus avec interactions n'a pas été étudiée à ma connaissance. Des techniques d'élagage d'arbres et de constructions lookdown ont été récemment développées dans les cadres avec interactions, voir par exemple Berestycki et al. [14], Le et al. [100], Blancas et al. [28], Etheridge et al. [55]. Il s'agit peut être d'une porte d'entrée pour comprendre ce que représente le processus dual U trajectoriellement pour Z.

— **Caractérisation de la dualité de Laplace**. Nous avons établi avec Matija Vidmar dans [F17] que les CSBPs avec collisions sont les seuls processus de Feller sans sauts négatifs en dualité de Laplace avec les diffusions non tuées. Il serait intéressant de comprendre plus généralement la structure des processus de Markov en dualité de Laplace.

Organisation des annexes

Le document comporte quatre annexes.

Dans l'Annexe A, nous rappelons des informations de base sur les processus de branchement en temps continu. En particulier les conditions d'explosion et d'extinction sont données, ainsi que la forme des générateurs et les constructions classiques (changement de temps, équation stochastique) des CSBPs.

Dans l'Annexe B, la définition des processus extrémaux unidimensionnels est rappelé ainsi que quelques unes de leurs propriétés fondamentales. Dans l'Annexe C, nous commençons par définir tous les termes décrivant les points frontières que nous rencontrerons. Des éléments de base sur les diffusions unidimensionnelles sont en suite données (problème martingale, mesure de vitesse et fonction d'échelle, classification de Feller et temps d'atteinte).

L'Annexe D porte sur la notion de H-dualité pour une fonction H générale. Les liens entre les semi-groupes en dualité, le renversement du temps et les générateurs y sont brièvement expliqués.

Part I

Genealogy of branching processes and Siegmund duality

In the first part of this document we will focus on the study of the continuous-state space branching processes (CSBPs) and their genealogy. We give here the most fundamental elements needed to state with ease the results. More background and details are gathered in the Annex A.

CSBPs are Markov processes $(X_t(z), t \geq 0)$ (where we have emphasized the initial value $X_0(z) = z \geq 0$), with state-space $[0, \infty]$, satisfying the branching property

$$\forall z, z' \in [0, \infty], \forall t \geq 0, \ X_t(z + z') \overset{d}{=} X_t^{(1)}(z) + X_t^{(2)}(z'), \tag{0.1}$$

where $X_t^{(1)}(z)$ and $X_t^{(2)}(z')$ are independent copies of $X_t(z)$ and $X_t(z')$ respectively. This identity, combined with the Markov property, entails the existence of a map $(t, x) \mapsto u_t(x)$, called cumulant, such that for all $x \geq 0$ and all $t, s \geq 0$

$$\mathbb{E}[e^{-xX_t(z)}] = e^{-zu_t(x)} \text{ and } u_{s+t}(x) = u_s \circ u_t(x). \tag{0.2}$$

Moreover, there exists a unique function Ψ of the *Lévy-Khintchine* form

$$\Psi(q) = -\lambda + \frac{\sigma^2}{2}q^2 + \gamma q + \int_0^{+\infty} \left(e^{-qx} - 1 + qx\mathbb{1}_{\{x \leq 1\}}\right) \pi(\mathrm{d}x) \tag{0.3}$$

with $\lambda \geq 0$, $\gamma \in \mathbb{R}$, $\sigma \geq 0$, and π a σ-finite measure carried on $(0, \infty)$ satisfying

$$\int_0^{+\infty} (1 \wedge h^2)\pi(\mathrm{d}h) < +\infty$$

such that the map $t \mapsto u_t(x)$ is the unique solution to the integral equation

$$\forall t \in [0, +\infty), \forall x \in (0, +\infty)/\{\varrho\}, \quad \int_{u_t(x)}^{x} \frac{\mathrm{d}z}{\Psi(z)} = t \tag{0.4}$$

where $\varrho = \inf\{z > 0, \Psi(z) \geq 0\} \in [0, +\infty]$. The function Ψ is referred as the *branching mechanism*. It completely characterizes the cumulant $(u_t(\cdot), t \geq 0)$ through the integral equation (0.4) and in turn determines the law of the CSBP X. The phenomena of extinction, survival and explosion and their study with the help of Ψ are briefly recalled during the course of the chapter.

In Chapter 1, based on [F7], we study the asymptotic properties of the genealogy, when times goes forward, of the random population whose size varies as a CSBP. The question addressed here can be formulated as follows: How does the population decline or grow? In the supercritical case, some individuals have an infinite line of descent, but are their progeny sizes evolving at the same rate, or do some have a drastically larger number of descendants? Similarly, when the population is extinguishing, i.e., its size approaches zero but never reaches it; see Annex A.2.4, are there individuals whose progeny take significantly longer to vanish compared to others? In other words, how are growth and decay organized from the perspective of individuals?

In Chapter 2, built on [F9] and [F15], we explore the genealogy backwards in time. We consider a branching population of infinite size at all times with founders arbitrarily old and follow the ancestral lineages of individuals extant at a given fixed time, say 0. The primary tool here is Siegmund duality. This notion matches with the definition of ancestors in the encoding of the population through a flow. Different points of view are then taken to describe the genealogy. Tracing the lineage of a given individual back in time gives a certain Markov process with negative jumps (this is the Siegmund dual).

Its semigroup and infinitesimal generator are described and the long-term and boundary behaviors classified. In order to track not only the location of the ancestor but also the mergings between the lineages, we introduce elementary (non-exchangeable) Markovian coalescent processes. Last, and in a very similar spirit as in Chapter 1, we study the almost sure long-term behavior of the ancestral lineages in the subcritical case and describe explicitly the ancestral partition (i.e. the set of genuine ancestors from time $-\infty$).

CHAPTER 1

Flow of CSBPs and Extremal Processes

Summary.

We introduce first in this chapter the notion of flow of continuous-state branching processes with two parameters. This family of processes, indexed by their initial value $z \in \mathbb{R}_+$ encodes a continuous-state population model of infinite size at all times, in which the notion of infinitesimal individual is made clear. The possible long-term behaviors of this flow are studied. They are characterized through subordinators and extremal processes. The latter arise in the case of supercritical processes with *infinite* mean and of subcritical processes with *infinite* variation. The jumps of these extremal processes are then interpreted as specific initial individuals whose progenies overwhelm the population. These individuals, which correspond to the records of a certain Poisson point process embedded in the flow, are called super-individuals. They radically increase the growth rate to $+\infty$ in the supercritical case, and slow down the rate of extinction in the subcritical one.

1.1 A continuous population model

If the notion of individual is clear in the discrete setting (one only has to label each initial individual by an integer and choose a way to encode their descendants, for instance by the lexicographical order or a monotone coupling), it is more challenging to make this notion clear in the continuous state space. We explain here the approach of stochastic flow designed by Bertoin and Le Gall in [21]. This will be of particular relevance when studying the growth or decay of the CSBP from an individual's point of view as well as when tracing the full genealogy.

The key observation is the reformulation of the branching property (0.1) as the infinite divisibility of the positive random variable $X_t(z)$ at any fixed time t along the variable z. This allows one to consider $X_t(z)$ as the value of some subordinator (i.e. an increasing positive Lévy process) at "time" z. The Laplace exponent of the latter is moreover nothing but the cumulant function $x \mapsto u_t(x)$ defined in (0.2). It comes then naturally to consider the following object.

Definition 1.A. *We call flow of CSBPs with branching mechanism Ψ, a collection of random variables $(X_t(z), z \geq 0, t \geq 0)$ such that*

i) for all $t \geq 0$ $(X_t(z), z \geq 0)$ is a càdlàg subordinator with Laplace exponent $x \mapsto u_t(x)$ i.e. for any $t \geq 0, z' > z$, conditionally on $X_t(z) < \infty$:

(a) $X_t(z') - X_t(z)$ is independent of $\sigma(X_t(y), y \leq z)$,

(b) $X_t(z') - X_t(z)$ has the same law as $X_t(z' - z)$.

ii) for any $z' \geq z$, $(X_t(z') - X_t(z), t \geq 0)$ is a CSBP(Ψ) started from $z' - z$, independent of $(X_t(z), t \geq 0)$.

Several constructions of the flow $(X_t(z), t \geq 0, t \geq 0)$ have been proposed. For instance, it can be constructed using the Kolmogorov extension theorem, as detailed in [21]; through stochastic equations, as discussed by Dawson and Li [43], see Annex A.2.1, or via a Poisson construction, as shown in Pitman and Yor [124] for the Feller diffusion, and later by Duquesne and Labbé [46] and Li [108] in the most general setting. We will return to the Poissonian construction shortly. Now, we explain Bertoin and Le Gall's seminal idea, which provides a framework for interpreting the flow as a genuine continuous population model.

Individuals are encoded with the positive real numbers $(0, \infty)$. For any fixed time t, the jump locations (i.e. the jump times but along the variable z) of the subordinator $z \mapsto X_t(z)$ are the initial individuals which have descendants at time t. More precisely the individual a living at time 0 has for descendant b at time t, if

$$X_t(a-) < b < X_t(a).$$

The jump interval $(X_t(a-), X_t(a)]$ represents thus all the descendants of a and $\Delta X_t(a) := X_t(a) - X_t(a-)$ is the size of the family of a at time t (in particular it is zero if a is not a jump of $X_t(\cdot)$).

As we shall see, for any fixed time $t > 0$, whether the subordinator $X_t(\cdot)$ has a drift, a finite lifetime or an infinite Lévy measure actually depends on the behavior of the CSBP $(X_t, t \geq 0)$. Write the function u_t in the Bernstein form

$$u_t(x) = \kappa_t + d_t x + \int_0^\infty (1 - e^{-xr}) \ell_t(\mathrm{d}r), \tag{1.1}$$

with $\kappa_t, d_t \geq 0$ and ℓ_t a Lévy measure on $(0, \infty)$ such that $\int_0^\infty (1 \wedge r) \ell_t(\mathrm{d}r) < \infty$.

When Grey's condition holds, i.e. $\int^{\infty} \frac{du}{|\Psi(u)|} < \infty$, the CSBP $(X_t(z), t \geq 0)$, started from any $z \in (0, \infty)$, gets extinct in finite time with positive probability, see Theorem A.22. The subordinator $X_t(\cdot)$ is in this case a compound Poisson process $(d_t = 0, \ell_t(0, \infty) < \infty)$, see Table 1.1 and Figure 1.1. Notice that the constant stretches represented in the figure correspond to initial individuals with no descendant at time t.

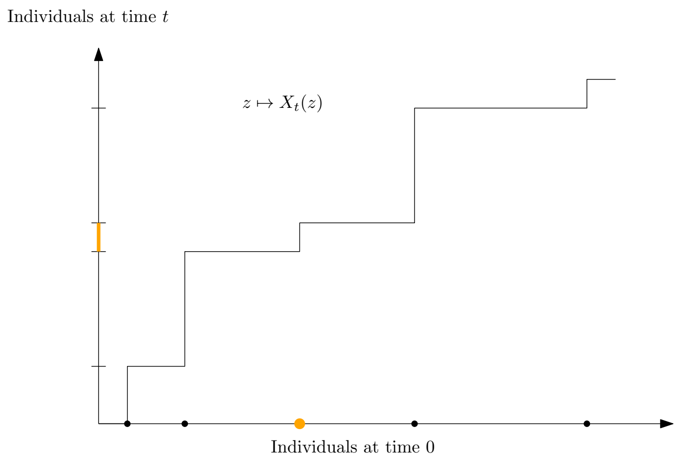


Figure 1.1 – Subordinator's encoding of the families at time $t > 0$ when $\int^{\infty} \frac{du}{|\Psi(u)|} < \infty$.

When Grey's condition does not hold, the CSBP is persistent (it does not touch 0) and the subordinator $X_t(\cdot)$ is not anymore a compound Poisson process. It has an infinite Lévy measure ℓ_t at any time $t > 0$ and initial individuals with descendants at time t form a dense subset of $(0, \infty)$. More precisely, without Grey's condition:

— in the infinite variation setting, $\Psi'(\infty) = \infty$, $X_t(\cdot)$ has no drift for any $t > 0$ and all individuals have progenies (potentially "microscopic") at any positive time.

— In the finite variation setting, $\Psi'(\infty) < \infty$, $X_t(\cdot)$ has a positive drift, i.e. $d_t > 0$, for any $t > 0$ and if $X_t(z) = X_t(z-)$, then the singleton $\{X_t(z)\}$ is the descendant at time t of z. In other words, the drift term encodes individuals alive at time t that have not yet started to reproduce and whose progeny is somehow reduced to themselves [1]).

We stress that there is no case for which $X_t(\cdot)$ is a compound Poisson process with a drift.

Last, when Dynkin's condition, i.e. $\int_0 \frac{du}{|\Psi(u)|} < \infty$ holds, the CSBP explodes (i.e. it hits and is absorbed at ∞), see Theorem A.22. This is necessary and sufficient for the subordinator $X_t(\cdot)$ to have a finite lifetime, hence $\kappa_t > 0$. In this case, the jump to ∞ in $X_t(\cdot)$ represents the first initial individual with an infinite progeny at time t.

We sum up what we just explained in the following table. See Annex A.2.6 for more details.

$\kappa_t = 0$	$u_t(0+) = 0$	Non-explosion $\int_0 \frac{du}{\lvert\Psi(u)\rvert} = \infty$
$d_t = e^{-\Psi'(\infty)t} = 0$	$\lim_{x\to\infty} u_t(x)/x = 0$	Infinite variation $\Psi'(\infty) = \infty$
$d_t = 0$ and $\ell_t(0, \infty) < \infty$	$u_t(\infty) < \infty$	Extinction $\int^{\infty} \frac{du}{\Psi(u)} < \infty$

Table 1.1 – Classification of the Lévy triplet of the cumulant u_t

1. We may also see those individuals as the *dust* in Fragmentation/coalescence theory

Poisson construction of the flow. One way to construct $(X_t(z), z \geq 0, t \geq 0)$ is to use a Poisson point process on the space of càdlàg trajectories. We explain it briefly in the case of infinite variation [2]. When $\Psi'(\infty) = \infty$, the cumulant function u_t is driftless, i.e. $d_t \equiv 0$ and the family of Lévy measure $(\ell_t, t > 0)$ on $\mathbb{R}^+ \cup \{+\infty\}$ forms an entrance law for the semigroup of the CSBP(Ψ) restricted to $(0, \infty]$. This yields the existence of a measure N_Ψ (called cluster measure in [46], excursion measure [3] in [109], canonical measure in [101]) on the space $\mathcal{D}$ of càdlàg paths from $\mathbb{R}_+^\star$ to $\mathbb{R}_+ \cup \{+\infty\}$ such that for any non-negative function F

$$N_\Psi(F(X_{t+\cdot}); X_t > 0) = \int_{(0,+\infty]} \ell_t(\mathrm{d}x)\mathbb{E}_x^\Psi(F) \text{ and } N_\Psi(X_{0+} > 0) = 0. \quad (1.2)$$

Setting for all $z \geq 0$ and $t > 0$,

$$X_t(z) = \sum_{z_i \leq z} X_t^i, \quad (1.3)$$

with $X_0(z) = z$ where $\mathcal{N} = \sum_{i \in I} \delta_{(x_i, X^i)}$ over $\mathbb{R}_+ \times \mathcal{D}$ is a PPP with intensity $\mathrm{d}x N_\Psi(\mathrm{d}X)$ and the set I denotes a countable enumeration of its atoms.

The flow $(X_t(z), z \geq 0, t \geq 0)$ satisfies the properties i) and ii) of Definition 1.A and for any $z \in \mathbb{R}_+$, $\Delta X_t(z) := X_t(z) - X_t(z-)$ represents the progeny at time t of the individual z at time 0. By the Poisson construction, any individual with a non zero progeny at some positive time belongs to $\{z_i, i \in I\}$.

Remark 1.1. Roughly speaking, we sum independent CSBPs with same mechanism Ψ, starting from individuals $(z_i, i \in I)$ with zero mass. One can interpret N_Ψ as the Lévy measure of the *path-valued* subordinator $(X_t(\cdot))_{t \geq 0}$, see Li [107].

We now introduce two particular types of individuals. Firstly, in a supercritical CSBP, some initial individuals have an infinite number of descendants, in other words, their family size at time t tends towards infinity as t goes to infinity. These individuals are called prolific and are in fact solely responsible for the infinite growth of the process. They were introduced and studied in the continuous state space framework by Bertoin et al. in [19].

Definition 1.2. The individual z is said to be prolific if $\lim_{t \to +\infty} \Delta X_t(z) = +\infty$. Denote by $\mathcal{P}$ the set of prolific individuals

$$\mathcal{P} := \{z > 0; \lim_{t \to +\infty} \Delta X_t(z) = +\infty\}.$$

We shall see that in a non-explosive CSBP with infinite mean (with or without infinite variation) and in a persistent CSBP with infinite variation, some individuals have a progeny that overwhelms the total progeny of all individuals *below* them (see Definition 1.3). We coin the name of *super-individuals* in [F7].

Definition 1.3 (Definition 2 in [F7])**.** The individual z is said to be a super-individual if $\lim_{t \to +\infty} \frac{\Delta X_t(z)}{X_t(z-)} = +\infty$ a.s. Denote by $\mathcal{S}$ the set of super-individuals

$$\mathcal{S} := \left\{ z > 0; \lim_{t \to +\infty} \frac{\Delta X_t(z)}{X_t(z-)} = +\infty \right\}. \quad (1.4)$$

2. For the finite variation case, see Duquesne and Labbé in [46] or Li's lecture notes [108] for a Poisson construction.

3. This is not however the classical Itô's measure since 0 is an exit boundary and $\mathbb{N}_\Psi(1 \wedge \zeta) = \infty$ where ζ is the excursion's length

We stress that there is an order between the super-individuals: if z_1 and z_2 are in $\mathcal{S}$ and $z_1 < z_2$ then the progeny of z_2 overwhelms that of z_1, since

$$0 \le \frac{\Delta X_t(z_1)}{\Delta X_t(z_2)} \le \frac{X_t(z_2-)}{\Delta X_t(z_2)} \underset{t\to+\infty}{\longrightarrow} 0.$$

In the supercritical case, we say that an individual is *super-prolific*, if it is a prolific super-individual. We will see that only certain prolific individuals are super-prolific.

In the subcritical case, since all initial individuals have progenies that get extinct (in finite time or not), no prolific individual may exist. However, when the process is persistent with infinite variation, super-individuals do exist. They are individuals whose progeny decays at a much slower rate than all individuals below them. In the finite variation case and finite mean case, $\mathcal{S}$ is degenerate (empty or reduced to a single point) and there are essentially no super-individuals.

1.2 Almost-sure renormalizations

The quest of finding a.s. renormalizations for branching processes has been undertaken by many authors first in discrete time and space and later in the continuous-state space setting. An obvious candidate for renormalizing the process is given by the *inverse* function of the cumulant $u_t(\cdot)$, denote it by u_{-t}, one has by (0.2)

$$\mathbb{E}_z[e^{-u_{-t}(x)X_t}] = e^{-zx}.$$

The function $x \mapsto u_{-t}(x)$ is well-defined on $[0, u_t(\infty))$, strictly increasing and satisfies for all $s, t \in \mathbb{R}^+$ that when $0 \le x < u_{s+t}(\infty)$,

$$u_{-(s+t)}(x) = u_{-s} \circ u_{-t}(x)$$

and for all $t \ge 0$ and $x < u_t(\infty)$

$$\int_{u_{-t}(x)}^{x} \frac{\mathrm{d}u}{\Psi(u)} = -t. \tag{1.5}$$

Last but not least, for all $z \ge 0$, and $x \in (0, \varrho)$, the process

$$(M_t^x(z), t \ge 0) := (\exp\left(-u_{-t}(x)X_t(z)\right), t \ge 0)$$

is a martingale (called Grey's martingale). For any fixed z, $M^x(z)$ being positive, it converges almost surely and $u_{-t}(x)X_t(z)$ admits an almost sure limit, call it $W^x(z)$, in $[0, \infty]$.

We observe next two different regimes: when the mean or the variation are finite, $u_{-t}(x)$ as t goes to ∞ remains of the same order for different values of x, that is to say

$$\frac{u_{-t}(x)}{u_{-t}(y)} \underset{t\to\infty}{\longrightarrow} c_{x,y} \in (0, \infty). \tag{1.6}$$

At the contrary, when the mean or the variation is infinite, changing x alters significantly the growth or decay of the function $t \mapsto u_{-t}(x)$. In this case, the function is rapidly varying: for example, in case of an infinite mean, i.e. $\Psi'(0+) = -\infty$, for any $x < y$

$$\frac{u_{-t}(x)}{u_{-t}(y)} \underset{t\to\infty}{\longrightarrow} 0. \tag{1.7}$$

Heuristically, by exchanging, the role of x with that of z, in a *dual* way, we will see that either all families have sizes of the same order (case of finite mean and finite variation), or certain individuals z are founding gigantic families (they will be the super-individuals). In the first setting, $W^x(z)$ is valued in $[0, \infty)$ for any z. In the latter, $W^x(z)$ is a degenerated random variable with values in $\{0, \infty\}$.

1.2.1 Finite mean and finite variation

The renormalized *flow* in the finite mean/variation setting was first studied by Duquesne and Labbé [46].

Theorem 2.A (Supercritical case with finite mean, Duquesne and Labbé [46]). *Assume Ψ supercritical with finite mean: $\Psi'(0+) \in (-\infty, 0)$. Let $x \in (0, \varrho)$. Almost-surely for all $z > 0$,*

$$u_{-t}(x) X_t(z) \underset{t \to +\infty}{\longrightarrow} W^x(z), \tag{1.8}$$

where W^x is a càdlàg subordinator with Laplace exponent $\theta \mapsto u_{\frac{\log(\theta)}{-\Psi'(0+)}}(x)$. Its Lévy measure has total mass $\varrho \in (0, +\infty]$. Moreover

$$\{X_t(z) \underset{t \to \infty}{\longrightarrow} 0\} = \{W^x(z) = 0\} \text{ and } \mathcal{P} = \{z^\star > 0 : W^x(z^\star) > W^x(z^\star -)\}, \tag{1.9}$$

and if $\varrho = +\infty$, then $\mathcal{S} \cap \mathcal{P} = \emptyset$ a.s.. If $\varrho < +\infty$, then $\mathcal{S} \cap \mathcal{P} = \{z_1^\star\}$ a.s., with $z_1^\star$ the first prolific individual.

Such almost sure renormalizations exist also in the framework of Bienaymé-Galton-Watson Markov chains. They are known as Seneta-Heyde norming, see e.g. Athreya-Ney's book [5]. The function $t \mapsto u_{-t}(x)$ must be then replaced by a positive sequence with an implicit definition. In the smoothier setting of continuous time and space, the function $t \to u_{-t}(x)$ simplifies the study in many aspects. In this spirit, Grey [68] has found the following condition on the branching mechanism for the growth to be *exactly* exponential (known as Kesten-Stigum theorem in the discrete world).

Proposition 2.A (LlogL condition). *When $\Psi'(0+) \in (-\infty, 0)$, for any $x > 0$, there exists a positive constant c_x such that*

$$u_{-t}(x) \underset{t \to \infty}{\sim} c_x e^{\Psi'(0+)t} \text{ if and only if } \mathcal{I} := \int_0 \left(\frac{1}{\Psi(u)} - \frac{1}{\Psi'(0+)u} \right) \mathrm{d}u < \infty.$$

Moreover $\mathcal{I} < \infty$ is equivalent to $\int^\infty h \log h \pi(\mathrm{d}h) < \infty$.

Remark 1.4. The renormalization of the flow in (1.8) being linear, the limiting process W^x is a subordinator. Its infinite divisibility property, to wit for any n

$$W^x(z) \overset{d}{=} W^1(z/n) + \ldots + W^n(z/n), \text{ with } W^i(z/n) := W^x(iz/n) - W^x((i-1)z/n) \text{ i.i.d}$$

mirrors the fact that the progeny of the individuals $[0, z]$ grows as the sum of individuals progenies in $[0, z]$. The identity (1.9) tells us that in the supercritical case, only prolific individuals participate to the growth.

Remark 1.5.

1. The parameter x in the renormalization $u_{-t}(x)$ and in the subordinators W^x does not play an important role since $u_{-t}(x) = c_{x,x'} u_{-t}(x')$ for some constant $c_{x,x'} \in (0, \infty)$.
2. The subordinator $(W^x(z), z \geq 0)$ is driftless.

As mentioned at the beginning of the chapter, a peculiar feature of the continuous-state space lies in the fact CSBPs might be persistent even in the (sub)critical cases (this stems to the fact that there exist branching mechanisms for which $\Psi(\infty) = \infty$ and $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} = \infty$). This covers two settings: either the process is of finite variations or not. In the former setting, the flow can also be linearly renormalized, see the subsequent theorem.

Theorem 2.B ((Sub)-critical case with finite variation: Duquesne and Labbé [46]). *Suppose $\Psi'(\infty) \in \mathbb{R}$ and fix $x \in (0, +\infty)$. Almost-surely for any $z > 0$,*

$$u_{-t}(x)X_t(z) \underset{t\to+\infty}{\longrightarrow} V^x(z),$$

where $(V^x(z), z \geq 0)$ is a càdlàg subordinator with Laplace exponent $\theta \mapsto u_{-\frac{\log\theta}{\Psi'(\infty)}}(x)$. Its Lévy measure has infinite mass and moreover $\mathcal{S} = \emptyset$ almost-surely.

It is furthermore established in [46] that V has a drift (i.e. there is "persistent" dust) if and only if $\int_0 h \log 1/h\pi(\mathrm{d}h) < \infty$. In a similar spirit as Proposition 2.16, the latter integral condition is also a necessary and sufficient for having $u_{-t}(x) \underset{t\to\infty}{\sim} c_x e^{\Psi'(\infty)t}$ for some constant $c_x > 0$. In such a case, the decay of the population size occurs exponentially fast.

1.2.2 Infinite mean or infinite variation

In the case of infinite mean or variation, linear renormalization is no longer suited, as family sizes do not evolve on the same scale. The renormalization will be non-linear and we introduce the following functions. Let $x_0 > 0$ be fixed.

— Assume Ψ with infinite mean and non-explosive, i.e. $\Psi'(0) = \infty$ and $\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} = \infty$. Recall $\varrho \in (0, \infty]$ the largest zero of Ψ. The map

$$G : y \mapsto \exp\left(-\int_y^{x_0} \frac{\mathrm{d}u}{\Psi(u)}\right),$$

is continuous, non-increasing, goes from $[0, \varrho]$ to $[0, +\infty]$ and is slowly varying at 0. One has moreover the following relation with u_{-t}

$$G(u_{-t}(x)) = G(x)e^t. \tag{1.10}$$

— Assume Ψ (sub)-critical, with infinite variation and persistent i.e. $\Psi'(\infty) = +\infty$ and $\int^{+\infty} \frac{\mathrm{d}u}{\Psi(u)} = +\infty$, for any $x_0 > 0$, the map

$$H : y \mapsto \exp\left(-\int_{x_0}^{y} \frac{\mathrm{d}u}{\Psi(u)}\right),$$

is continuous, non-increasing, goes from $[0, +\infty]$ to $[0, +\infty]$ and is slowly varying at $+\infty$. Similarly, H is related to u_{-t} as follows

$$H(u_{-t}(x)) = H(x)e^{-t}. \tag{1.11}$$

Theorem 1.6 (Theorem 1 in [F7]).

i) Assume Ψ supercritical with infinite mean and non-explosive. Almost surely for all $z \geq 0$

$$e^{-t}G\left(\frac{1}{X_t(z)} \wedge \varrho\right) \underset{t\to+\infty}{\longrightarrow} R(z) := \sup_{z_i^\star \leq z} R_i,$$

with

$$\mathcal{M}_G := \sum_{i\in I} \delta_{(z_i^\star, R_i)} \text{ a Poisson point process over } \mathbb{R}_+ \times (0, \infty)$$

with intensity $\mathrm{d}z \otimes \mu_G(\mathrm{d}r)$ such that $\overline{\mu}_G(r) := u_{\log(\frac{1}{r})}(x_0)$. Moreover,

$$\{X_t(z) \underset{t\to\infty}{\longrightarrow} 0\} = \{R(z) = 0\} \text{ and } \mathcal{S} \cap \mathcal{P} = \{z > 0 : \Delta R(z) > 0\} \text{ a.s.}$$

ii) Assume Ψ (sub)-critical with infinite variation and persistent. Almost-surely for all $z \geq 0$

$$e^t H\left(\frac{1}{X_t(z)}\right) \underset{t\to+\infty}{\longrightarrow} Q(z) := \sup_{z_i \leq z} Q_i,$$

with

$$\mathcal{M}_H := \sum_{i\in I} \delta_{(z_i,Q_i)} \text{ a Poisson point process over } \mathbb{R}_+ \times (0,\infty)$$

with intensity $\mathrm{d}z \otimes \mu_H(\mathrm{d}r)$ *such that* $\overline{\mu}_H(r) = u_{\log r}(x_0)$. *Moreover,*

$$\mathcal{S} = \{z > 0 : \Delta Q(z) > 0\} \text{ a.s.}$$

The Poisson point processes $\mathcal{M}_G$ and $\mathcal{M}_H$ represent the initial individuals with their asymptotic growth and decay rates respectively. In the supercritical case, on the event $\{X_t(z) \underset{t\to\infty}{\longrightarrow} 0\}$, $\frac{1}{X_t(z)} \wedge \varrho$ tends towards ϱ, and since $G(\varrho) = 0$, one has $Q(z) = 0$. This explains that the non-prolific individuals do not appear in $\mathcal{M}_G$. In both the subcritical case and supercritical immortal case (i.e. $\varrho = \infty$) the intensity measure of the PPP is infinite and there are infinitely many super-individuals near zero (the processes R and Q jump instantaneously away from zero).

Remark 1.7. The limiting processes R and Q, defined as the records of a PPP, belong to the class of the so-called *extremal process*. In particular, they satisfy a *max-infinite divisibility* property. Focussing on R, the random variable $R(z)$ has the same law as the maximum of n independent copies of $R(z/n)$:

$$R(z) \overset{d}{=} \max(R^1(z/n), ..., R^n(z/n)).$$

Heuristically, the infinite mean of the branching law transforms the sum, see Remark 1.4, into a maximum.

Sketch of the proof of Theorem 1.6. We focus on the supercritical case and assume $\Psi'(0+) = -\infty$. Grey's martingale provides that

$$u_{-t}(x)X_t(z) \underset{t\to+\infty}{\longrightarrow} W^x(z) := \begin{cases} 0 & \text{with probability } e^{-xz}, \\ \infty & \text{with probability } 1 - e^{-xz}. \end{cases} \tag{1.12}$$

Step 1. is to get the almost-sure convergence for a fixed initial value z. This was done by Grey [69] in a discrete setting. The random function $\{W^x(z) : z \in (0,\varrho) \cap \mathbb{Q}\}$ steps up from 0 to $+\infty$ at the random threshold $\Lambda_z := \inf\{x \in (0,\varrho) \cap \mathbb{Q} : W^x(z) = +\infty\} \in \mathbb{R}_+$ and is otherwise constant. The latter random variable is exponentially distributed with parameter z on $(0,\varrho)$ with a mass at ϱ: that is to say $\mathbb{P}(\Lambda_z \leq x) = 1 - e^{-zx}$ for $\lambda \in [0,\varrho)$ and $\mathbb{P}(\Lambda_z = \varrho) = e^{-z\varrho}$. Moreover, one can check that $\{\Lambda_z = \varrho\} = \{X_t(z) \underset{t\to+\infty}{\longrightarrow} 0\}$ a.s.

Assume $\Lambda_z < \varrho$. Choose x' and x'' such that $x', x'' \in (0,\varrho) \cap \mathbb{Q}$ and $x' < \Lambda_z < x''$, if t is large enough,

$$u_{-t}(x'')X_t(z) \geq 1 \text{ and } u_{-t}(x')X_t(z) \leq 1,$$

thus, together with the identity (1.10), we get

$$G(x') = e^{-t}G(u_{-t}(x')) \geq e^{-t}G(1/X_t(z)) \geq e^{-t}G(u_{-t}(x'')) = G(x'').$$

Since x' and x'' are arbitrarily close to Λ_z, and G is continuous, we obtain

$$e^{-t}G\left(\frac{1}{X_t(z)}\right) \underset{t\to+\infty}{\longrightarrow} G(\Lambda_z) \quad \mathbb{P}\text{-almost surely on } \{\Lambda_z < \varrho\}.$$

It remains now to understand the collection of limiting random variables $G(\Lambda_z)$ as a process indexed by z. We sketch the arguments. This is done in two steps.

Step 2. Define for any $z \in \mathbb{Q}_+$, $\widetilde{R}(z) := G(\Lambda_z)$. We content ourselves to show how the sum is transformed into a maximum along a calculation for the two-dimensional marginal law of $(\widetilde{R}(z), z \in \mathbb{Q}_+)$. Let $y_1, y_2 \in \mathbb{R}_+$ and set $\lambda_1 = G^{-1}(y_1)$ and $\lambda_2 = G^{-1}(y_2)$. By construction the events $\{\widetilde{R}(z_1) < y_1, \widetilde{R}(z_2) < y_2\}$ and $\{W^{\lambda_1}(z_1) = 0, W^{\lambda_2}(z_2) = 0\}$ are identical. Then by (1.12), the branching and Markov properties

$$\begin{aligned}
&\mathbb{P}(\widetilde{R}(z_1) < y_1, \widetilde{R}(z_2) < y_2)\\
&= \mathbb{P}(W^{\lambda_1}(z_1) = 0, W^{\lambda_2}(z_2) = 0)\\
&= \lim_{t\to+\infty} \mathbb{E}\left[\exp\Big(-u_{-t}(\lambda_1)X_t(z_1) - u_{-t}(\lambda_2)X_t(z_2)\Big)\right]\\
&= \lim_{t\to+\infty} \exp\left(-z_1 u_t(u_{-t}(\lambda_1) + u_{-t}(\lambda_2))\right)\exp\left(-(z_2 - z_1)u_t(u_{-t}(\lambda_2))\right)\\
&= e^{-z_1 G^{-1}(y_1)\vee G^{-1}(y_2)} e^{-(z_2-z_1)G^{-1}(y_2)} = e^{-z_1 G^{-1}(y_1\wedge y_2)} e^{-(z_2-z_1)G^{-1}(y_2)},
\end{aligned}$$

where in the penultimate equality, the second exponential term is directly obtained since by definition $u_t(u_{-t}(\lambda_2)) = \lambda_2 = G^{-1}(y_2)$ and the first term comes from the fact that in the infinite mean case, by (1.7),

$$u_t(u_{-t}(\lambda_1) + u_{-t}(\lambda_2)) \underset{t\to+\infty}{\longrightarrow} \lambda_1 \vee \lambda_2.$$

We recognize here the two-dimensionals law of an extremal process based on the measure μ with tail $\overline{\mu}(y) = G^{-1}(y)$ for all $y \geq 0$, see Annex B and e.g. Resnick's book [126].

Step 3: Canonical constructions of extremal processes are derived from the records of a Poisson point processes, see Annex B. We aim now to identify the extremal process, obtained at the limit, within the Poissonian construction of the flow, establishing the almost sure convergence for all z and understanding that it "jumps exactly on the set of super-individuals".

By applying the almost sure convergence result of Step 1 under the cluster measure to each atom $X^i_{s+\cdot}$ in (1.3) viewed from a time $s > 0$ such that $X^i_s > 0$, we get by letting s go to 0 in a second time, the Poisson point process $\mathcal{M}_G$. The previous result shows that taking the sum in the prelimit leads to the maximum. It remains to compare the extremal process R obtained from $\mathcal{M}_G$ with the process $\widetilde{R}$ (defined on the rationals). The sample paths of R being càdlàg and the convergence towards $\widetilde{R}$ holding on a dense subset of individuals, one can conclude the convergence for all z. The correspondence between the jump times of R and the super-individuals is deduced from the slow variation of G. To simplify the discussion, consider the case $G(1/y) \sim \log y$, which implies that the growth of a prolific individual, say z_i, is approximatively $e^{R_i e^t}$. When the extremal process R jumps, at, say, $z^\star$, a new record of $\mathcal{M}_G$ is found. The population coming from individuals in $[0, z^\star)$ grows at the rate $e^{R(z^\star-)e^t}$, which is negligible compared to that of the family founded by $z^\star$ whose size is of order $e^{R(z_\star)e^t}$.

To achieve the proof, it is not difficult to check from (1.10) that $\overline{\mu}(y) = G^{-1}(y) = u_{\log(1/y)}(x_0)$. In particular, the total mass of μ is ϱ. Notice that when $\varrho = \infty$, i.e. the CSBP is immortal, see Annex A.2.5, there are infinitely many super-individuals close to 0 and the extremal process R has its state 0 instantaneous. □

We state below some conditions on the branching mechanism for the growth of the process to be double-exponential.

Proposition 1.8 (Propositions 2 and 5 in [F7]).

— *Assume* $\Psi'(0+) = -\infty$, $\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} = \infty$. *If* $\int_0 \left|\frac{1}{\Psi(u)} - \frac{1}{\alpha u \log u}\right| \mathrm{d}u < +\infty$, *for some* $\alpha > 0$ *then,*

$$e^{-\alpha t} \log\left(X_t(z)\right) \underset{t\to+\infty}{\longrightarrow} kR(z)^{\alpha} \ a.s.,$$

for some constant $k > 0$ *(which does not depend from* z*).*

— *Assume* $\Psi'(\infty) = +\infty$ *and* $\int^{+\infty} \frac{\mathrm{d}u}{|\Psi(u)|} = +\infty$. *If* $\int^{+\infty} \left|\frac{1}{\Psi(u)} - \frac{1}{\alpha u \log u}\right| \mathrm{d}u < +\infty$ *for some* $\alpha > 0$, *then*

$$e^{-\alpha t} \log\left(1/X_t(z)\right) \underset{t\to+\infty}{\longrightarrow} k' Q(z)^{-\alpha} \ a.s.,$$

for some constant $k' > 0$ *(which does not depend from* z*).*

In the Neveu case, i.e. when $\Psi(q) = q \log q$, see (A.24), the conditions of the above proposition are satisfied with $\alpha = 1$. Moreover by a symmetry property of the Cauchy distribution, the Neveu CSBP can be renormalized with the same function on both events of extinction and non-extinction.

Proposition 1.9 (Neveu flow, Proposition 7 in [F7]). *Consider* $(X_t(z), t \geq 0, z \geq 0)$ *a flow of Neveu CSBPs (constructed as in (1.3)). Then almost-surely for all* $i \in I$, *the limit* $R_i := \lim_{t\to+\infty} e^{-t} \log X_t^i$ *exists. The point process* $\mathcal{M} := \sum_{i\in I} \delta_{(z_i, R_i)}$ *is a Poisson point process over* $\mathbb{R}_+ \times \mathbb{R}$ *with intensity* $\mathrm{d}z \otimes e^{-r}\mathrm{d}r$ *and almost-surely, for all* $z \geq 0$,

$$e^{-t} \log X_t(z) \underset{t\to+\infty}{\longrightarrow} R(z) := \sup_{z_i \leq z} R_i. \tag{1.13}$$

Moreover $\mathcal{S} = \{z > 0; \Delta R(z) > 0\}$ *a.s.*

The CSBPs of Neveu and its limit (1.13) (for a fixed z) have been used in the study of Derrida's random energy model by Bovier and Kourkova [33] and Huillet [77]. We refer also the reader to Fleischmann and Sturm [62] and Fleischmann and Wachtel [63] where superprocesses with Neveu's branching mechanism are studied and the limit (1.13) is discussed. The almost-sure convergence for a fixed z in Proposition 1.9 was first established by Neveu in the (unpublished) work [117].

We have restricted our attention to the non-explosive and persistent cases. Extremal processes arise nevertheless also easily in CSBPs that get absorbed in 0 or ∞ in finite time through the explosion and extinction times indexed by the initial states.

To conclude this chapter, we explain the link between the results above and some obtained by Duquesne and Labbé in [46]. They were interested in the phenomenon, called *Eve property*, for which the population concentrates on the progeny of a single individual, see also Bertoin's book [18, Chapter 4.4, page 205].

Definition 2.A. *The population started from a fixed size* z *has the Eve property if there exists a random variable* $\mathsf{e} \in [0, z]$, *called the Eve, such that*

$$\frac{\Delta X_t(\mathsf{e})}{X_t(z)} \underset{t\to\zeta_z}{\longrightarrow} 1 \ a.s., \tag{1.14}$$

where $\zeta_z := \inf\{t \geq 0; X_t(z) \in \{0, +\infty\}\} \in \mathbb{R}_+ \cup \{+\infty\}$.

The Eve property holds precisely when extremal processes arise (i.e. in the infinite variation or infinite mean case). The following statement is a corollary of Theorem 1.6. It was proven along different arguments in [46].

Corollary 1.10 (Theorem 0.3-(ii-a)-(iii-a) in [46] and [F7]). *For any fixed initial size, there exists an Eve if and only if*

$$\Psi'(0) = -\infty \text{ or } \Psi'(\infty) = \infty.$$

From (1.4) and (1.14), we see that the Eve e must be a super-individual whose progeny overwhelms those of individuals in $(\mathsf{e}, z]$. In our setting, the Eve of the population started with size z is therefore characterized as the *last* super-individual in $[0, z]$. In other words, the Eve is the individual in $[0, z]$ whose growth is the fastest in the non-explosive supercritical case with infinite mean or whose decay is the slowest in the persistent subcritical one with infinite variation.

1.3 Comments

Bertoin et al. in [19] have shown that the process counting the prolific individuals, when time evolves, is an *immortal* branching process. By immortal it is meant that no individual dies so that the reproduction law gives mass only to $\mathbb{N} := \{1, 2, \cdots\}$, see Annex A.1.2. This discrete process is related to the backbone decomposition which has been deeply studied by Berestycki et al. [15], Kyprianou et al. [90] and by Duquesne and Winkel [48] in the framework of random trees. We shall encounter this process in the next chapter. The process following the number of super-individuals along time has not been studied. Recalling however the classical fact that the number of records of a PPP on $[0, z]$ is of the order $\log z$, heuristically, the number of super-individuals issued from $[0, z]$ at time s evolves as $\log X_s(z)$ and thus, by Proposition 1.8, has typically an exponential growth.

On the setting with immigration

We saw in this chapter that when the branching law has no first moment, extremal processes come naturally into play after some non-linear renormalizations. Some reminiscent phenomena can be observed in the setting of a CSBP with immigration (CBI). In the latter, additionally to the branching, individuals arrive in the population, from an outside source, along the jumps and the drift of an independent subordinator with a given Laplace exponent Φ. In the same fashion as what has been done in this chapter for the pure branching processes, one may seek a renormalization of the CBIs.

It is known that with a log-moment on the immigration measure ν, a CBI, say, $(Y_t, t \geq 0)$, with a subcritical branching mechanism, i.e. $\Psi'(0+) > 0$, admits a limiting distribution, see e.g. Keller-Ressel and Mijatovic [84] and the references therein. Early in the seventies, Pinsky [122] initiated the study of CBIs outside the stationary setting and realized that their limiting behaviors were different according to the asymptotics of the integral $I(\varepsilon) := \int_\varepsilon^1 \frac{\Phi(u)}{u} \mathrm{d}u$ as ε goes to 0. We stress that $I(0+) < \infty$ if and only if ν has a log-moment.

With Chunhua Ma and Linglong Yuan, [F14], we have continued Pinsky's investigation and studied how to renormalise in law a CBI without limiting distribution.

Contrary to the setting of this chapter, we focused there on the case with finite mean branching and allow the immigration to be very active (with no log-moment). We found three regimes, defined according to the divergence's rate of $I(\varepsilon)$, leading to different limiting laws. In two particular regimes, the limiting laws were classical extremal laws. It turns out that in these regimes, the renormalization function is expressed with the help of the Laplace exponent Φ. More precisely, we have shown in [F14] a convergence in *law* of the following form:

$$\frac{1}{t\Phi(1/Y_t)} \underset{t\to\infty}{\Longrightarrow} M_1$$

where M_1 is a positive random variable with some explicit law.

We explore with Linglong Yuan further this convergence in [F19] by establishing a *functional* limit theorem for CBIs whose branching law has finite mean and immigration has (sub)-log tails. Assume that $x \mapsto x\Phi(e^{-x})$ is regularly (possibly slowly) varying at ∞ and admits an increasing equivalent function, then

$$\left(\frac{1}{t\Phi(1/Y_{st})}, s \geq 0\right) \underset{t\to\infty}{\Longrightarrow} (M_s, s \geq 0) \tag{1.15}$$

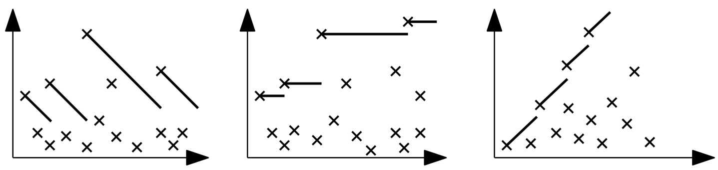
where M is a certain positive Markov process called *extremal shot noise process* (ESNs).

ESNs are Markov processes generalizing extremal processes (for which recalls are in Annex B), by adding a drift mechanism. They are defined as follows: for all time $s \geq 0$

$$M_s := \sup_{t\leq s}(R_t - b(s-t))_+$$

where $\mathcal{M} := \sum_{t\geq 0} \delta_{(t,R_t)}$ is a Poisson point process of intensity $\mathrm{d}t \otimes \mu(\mathrm{d}r)$, with μ such that $\overline{\mu}(x) < \infty$ for any $x > 0$ and $b \in \mathbb{R}$.

ESN with slope $-b$



$b > 0$ $b = 0$ $b < 0$

Figure 1.2 – Sample paths of an ESN(b, μ).

The ESN process arising in (1.15) has for parameters $b := \Psi'(0+)/c$ and $\mu(\mathrm{d}r) := \frac{\mathrm{d}r}{r^2}$ where the constant is defined as $c := \lim_{x\to\infty} x\Phi(e^{-x})$, which exists in $(0, \infty]$ by assumption.

In the case $c = \infty$, the drift of the ESN vanishes and we have a classical extremal process. In the case $0 < c < \infty$, one has $\Phi(1/y) \sim \frac{c}{\log(1+y)}$, as y goes to ∞, and (1.15) can be rewritten as

$$\left(\frac{1}{t}\log(1 + Y_{st}), s \geq 0\right) \Longrightarrow (\widetilde{M}_s, s \geq 0)$$

where $(\widetilde{M}_s, s \geq 0) := (\frac{1}{c}M_s, s \geq 0)$ is an ESN process. We recover in this setting a result of Iksanov and Kabluchko in [78] for Bienaymé-Galton-Watson processes with immigration.

This functional limit theorem sheds some light on the fact that the law of M_1, obtained in [F14], belongs to the extremal ones. Heuristically and in a very close spirit as the super-individuals, the limiting process M jumps at the times of immigration where the amount of arriving individuals changes radically the growth rate of the process in the log scale. When $0 < c < \infty$, there is an interplay between the growth of the branching dynamics, controlled by $\Psi'(0+)$, and the immigration. In the case $c = \infty$, the reproduction has no time to dictate the growth and the latter is entirely governed by the immigration subordinator.

ESNs and CBIs as Markov processes

A remarkable feature of ESN processes is that a direct and explicit study of their resolvent, semigroup, generator and local time at zero is possible. In particular, a core for the generator of ESNs is exhibited. The Markovian study of ESNs is done in [F18]. The proof of (1.15) in [F19], is made by showing uniform convergence of the generators on a core. Iksanov and Kabluchko [78] take a different route. Loosely speaking, they establish the convergence of the Poisson point processes "behind" the CBI Y and the ESN M.

As CBIs and ESNs will not be studied further in this text, here is perhaps a good place to mention very briefly the works [F5], [F6]. There is a very strong parallel between these two classes of processes. We mention for instance that CBIs are infinitely divisible Markov processes, while ESNs are *max*-infinitely divisible Markov processes.

In [F5], with Gerónimo Uribe Bravo, we studied the zero-sets of CBIs through a direct application of an important theorem about random covering of the half-line due to Fitzsimmons, Fristedt and Shepp, see [61].

In [F6] (respectively [F18]), the first passage times of the CBIs (resp. ESNs) below a level are studied. Criteria have been also designed to classify the transience/recurrence for both processes. We shall address this classical theme of Markov process theory in Chapter 4 for a class of processes called CSBPs with collision.

Last but not least, we wish to mention that the study of the local time at 0 of ESNs in [F18] enabled us to give a new proof of the Fitzsimmons-Fristedt-Shepp theorem using classical arguments from the theory of Markov processes.

CHAPTER 2

Backward genealogy of CSBPs, inverse flow and Siegmund duality

Summary.

We start by introducing Bertoin and Le Gall's flow of subordinators. The latter allows one to trace back the genealogy of the population between two arbitrary times $s < t$. Inverting the subordinators and reversing time give rise to a flow of coalescing Markov processes with negative jumps, which correspond to the ancestral lineages of individuals in the current generation. The process of the ancestral lineage of a fixed individual is the Siegmund dual process of the continuous-state branching process. We study its semigroup, its long-term behaviour and its generator. In order to follow the coalescences in the ancestral lineages and to describe the backward genealogy of the population, we define non-exchangeable Markovian coalescent processes obtained by sampling individuals according to an independent Poisson point process over the flow. These coalescent processes are called consecutive coalescents, as only consecutive blocks can merge. They are characterized in law by finite measures on $\mathbb{N}$ which can be thought as the offspring distributions of some inhomogeneous immortal Galton-Watson processes forward in time.

2.1 Introduction

Brief review of the literature

Both representations of genealogies with trees and subordinators are future-oriented and for a while not so much attention has been paid to the description of coalescences in ancestral lineages of branching processes (especially those in continuous-state space). We briefly review some methods that have been developed in order to study the genealogy backwards in time of branching processes.

As mentioned previously, when reproduction laws are stable, branching and resampling population models can be related through renormalisation by the total size and random time-change. We refer to Berestycki et al. [13], Birkner et al. [27], Foucart and Hénard [F2] and Schweinsberg [132]. The connection between exchangeable coalescents and CSBPs is particular to stable laws and the study of the genealogy of a general branching process requires a different method.

One approach consists in conditioning the process to be non-extinct at a given time, sampling two or more individuals uniformly in the population and study the time of coalescence of their ancestral lineages. This program is at the heart of the works of Athreya [4], Duquesne and Labbé[1] [46], Harris et al. [72], Johnston [81], Lambert [91] and Le [99].

Starting from a different point of view, Bi and Delmas [24] and Chen and Delmas [38] have considered stationary subcritical branching populations obtained as processes conditioned on the non-extinction. The genealogy is then studied via a Poisson representation of the population. We refer also to Evans and Ralph [57] for a study in the same spirit.

A third approach is to represent the backwards genealogy through point processes. Aldous and Popovic [2] and Popovic [125] have shown how to encode the genealogy of a critical Feller diffusion with a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+$ called *Coalescent Point Process*. Atoms of the coalescent point process represent times of coalescences between two "consecutive" individuals in the boundary of the Brownian tree. Such a description was further developed by Lambert and Popovic [94] for a Lévy continuum tree. In this general setting, multiple coalescences are possible and the authors build a point process with multiplicities, which records both the coalescence times and the number of involved mergers in the families of the current population. Their method requires in particular to work with the height process introduced by Le Gall and Le Jan in [102].

The approach of flows of subordinators

In the present chapter, based on the articles [F9] and [F15] written with Chunhua Ma, Bastien Mallein and Martin Möhle respectively, we choose a different route than those explained above and seek a dynamical description of the genealogy. We first observe that flows of subordinators provide a continuous branching population whose size is infinite at any time and whose ancestors are arbitrarily old. We then define and study the inverse flow, denoted by $(\widehat{X}_{s,t}(x), s \leq t, x \geq 0)$, which tracks backward in time the ancestral lineage of an individual considered at any given time. In particular, the process $(\widehat{X}_t(x), t \geq 0) := (\widehat{X}_{0,t}(x), t \geq 0)$ is the ancestral lineage of the individual x in the population taken at time 0. This is a Markov process and we characterize its semigroup, its long-term behaviour (recurrent or transient) as well as its generator.

1. their proof of the Eve property, see Corollary 1.10 in the previous chapter, was merely based on showing that the Eve is a common ancestor

In a second time, we introduce new elementary non-exchangeable Markovian coalescents as simple dual objects of immortal continuous-time Galton-Watson processes. These processes are taking values in the set of partitions of $\mathbb{N}$ whose blocks are formed with consecutive integers. We coin the name *consecutive coalescents* as only consecutive blocks will be allowed to merge. These coalescent processes represent the genealogy of immortal continuous-time Galton-Watson processes when time's arrow points to the past.

We will use these coalescent processes to describe the genealogy of general CSBPs. They will simplify the description given by the Coalescent Point Process as introduced in [94, Section 4]. Our method follows closely that of Bertoin and Le Gall for exchangeable coalescents [20], [22], [23]. Heuristically, exchangeable bridges are replaced by subordinators and uniform random variables by atoms of a Poisson process. We shall construct random partitions by sampling individuals according to an independent Poisson process. Namely, let $(J_i^\lambda, i \geq 0)$ be the sequence of atoms (i.e jumping times) of an independent Poisson process with intensity λ and consider, for any time $t \geq 0$, the random partition $C^\lambda(t)$ defined by letting integers i and j in the same equivalence class if and only if $\widehat{X}_t(J_i^\lambda) = \widehat{X}_t(J_j^\lambda)$. We will show that the process $(C^\lambda(t), t \geq 0)$ is a (possibly time-inhomogeneous) consecutive coalescent. We characterize its jump rates and its long-term behaviour.

We shall also show how to define the complete genealogy of individuals standing in the current generation when Grey's condition is satisfied. Loosely speaking, we will let the intensity parameter λ go to infinity and describe the genealogy through coalescing consecutive intervals of $[0, \infty]$.

In the Feller diffusion case, the study of the flow is particular. Coalescences and branching in this setting follows somehow the same dynamics and the inverse flow is a Feller diffusion with constant immigration. This is reminiscent to a reversal property of the Brownian tree, see Abraham and Delmas [1] and Delmas [44].

The chapter is organized as follows. In Section 2.2, we recall Bertoin and Le Gall's idea on how Bochner's subordination can be used to provide a representation of the genealogical structure associated with CSBPs. In Section 2.3, we investigate the inverse flow by characterizing its semigroup and studying its transience and recurrence. In Section 2.3.2, we describe the inverse flow in the case of the Feller diffusion. This also provides with an elementary approach the Coalescent Point Process of Popovic [125]. In Section 2.3.3, we investigate the martingale problem solved by the process of the ancestral lineage. In Section 2.4, we study the coalescences in the inverse flow of a general CSBP by defining the consecutive coalescents. We describe the genealogy of the whole population standing at the current generation under Grey's condition (recalled in Annex A). In Section 2.5, we study some fine properties of the long-term behaviors of the ancestral lineages in the subcritical case. This can be compared with the study in Chapter 1.

Last, we mention the recent work of Lambert and Johnston [80] where the coalescent structure of multi-dimensional CSBPs is studied. The techniques there are also based on Poissonian sampling and were developed independently. Besides the multi-dimensional setting, they also explain how to sample uniformly individuals in a finite population. We shall not work in this direction but rather consider an infinite population at all times.

2.2 Flows of subordinators

We have seen in the previous chapter, how to construct a two-parameter flow of CSBPs $(X_t(z), t \geq 0, z \geq 0)$ indexed by the initial value. This construction enables us to track the descendants of the initial individuals. Recall in particular that $(X_t(z), z \geq 0)$ is a subordinator (possibly killed) with Laplace exponent $x \mapsto u_t(x)$, see (0.2) and (1.1).

However this flow does not encode the family relationships of individuals seen between different generations, say s and t with $s < t$. Bertoin and Le Gall introduce in [21] a three-parameter flow $(X_{s,t}(z), s \leq t, z \geq 0)$ precisely for defining a complete genealogy. The idea lies in Bochner's subordination and we explain it below.

The semigroup property of the CSBP entails that for any $s, t \geq 0$,

$$u_{t+s} = u_t \circ u_s. \tag{2.1}$$

Bochner's subordination implies that if $Y^{(t)}$ is a subordinator with Laplace exponent u_t and $Y^{(s)}$ is an independent subordinator with Laplace exponent u_s, then $Y^{(t)} \circ Y^{(s)}$ is a subordinator with Laplace exponent $u_t \circ u_s = u_{t+s}$. Therefore, writing $\widetilde{X}$ an independent copy of the CSBP X, we have

$$\forall z \geq 0,\ X_{t+s}(z) \overset{(d)}{=} \widetilde{X}_t(X_s(z)). \tag{2.2}$$

This last observation led Bertoin and Le Gall [21] to consider representing a CSBP as a flow of subordinators, which we now define.

Definition 2.A. *A flow of subordinators is a family $(X_{s,t}(z), s \leq t, z \geq 0)$ satisfying the following properties:*

1. *For every $s \leq t$, $x \mapsto X_{s,t}(z)$ is a càdlàg subordinator, with same law as $z \mapsto X_{0,t-s}(z)$.*
2. *For every $t \in \mathbb{R}$, $(X_{r,s}, r \leq s \leq t)$ and $(X_{r,s}, t \leq r \leq s)$ are independent.*
3. *For every $r \leq s \leq t$, $X_{r,t} = X_{s,t} \circ X_{r,s}$.*
4. *For every $s \in \mathbb{R}$ and $x \geq 0$, we have $X_{s,s}(z) = z = \lim_{t \to s} X_{s,t}(z)$ in probability.*

It was proved by Bertoin and Le Gall [21] that any CSBP can be constructed as a flow of subordinators. Conversely for any flow, the process $(X_{s,t}(z), t \geq s)$ has the same finite-dimensional laws as a CSBP(Ψ) for a certain branching mechanism Ψ. The existence of such a flow is shown [2] in [21] by applying Kolmogorov extension theorem to a compatible family of laws on the Skorohod space of càdlàg paths. In particular, for any finite collection of times $s_1 < s_2 < \cdots < s_n$,

$$X_{s_1,s_n} = X_{s_{n-1},s_n} \circ \cdots \circ X_{s_1,s_2} \text{ a.s.} \tag{2.3}$$

and the processes $X_{s_{i-1},s_i}(\cdot)$ are càdlàg. We stress also that the parameters of time are in the whole [3] line $\mathbb{R}$.

In a similar fashion as what we have explained in Chapter 1, but with now tracking the individuals at all times, and not only the initial ones, the interval $[0, X_{s,t}(z)]$ can be interpreted as the set of descendants at time t of the population that was represented at time s by the interval $[0, z]$.

2. Another possibility to construct the flow of subordinators is to use a stochastic equation approach, see (A.17) in Annex A and the forthcoming Section 2.3.3.

3. To construct such a stationary flow on $\mathbb{R}$, one can start by defining it in $[-T, \infty)$ and argue by compatibility to let T go to ∞.

With this interpretation, the genealogy forward in time of the population is defined as follows. If $X_{s,t}(y-) < X_{s,t}(y)$, we say that for all $z \in (X_{s,t}(y-), X_{s,t}(y)]$, the individual z at time t is a descendant of the individual y living at time s. If $X_{s,t}(y-) = X_{s,t}(y)$ (i.e. $X_{s,t}$ is continuous at y) and $y = \inf\{x > 0 : X_{s,t}(x) = z\}$, we then say that individual $z = X_{s,t}(y)$ at time t is the descendant of the individual y living at time s.

One can observe that the flow property, i.e. $X_{r,t} = X_{s,t} \circ X_{r,s}$, ensures that this construction indeed defines a coherent genealogy in the sense that "children of children are grand-children".

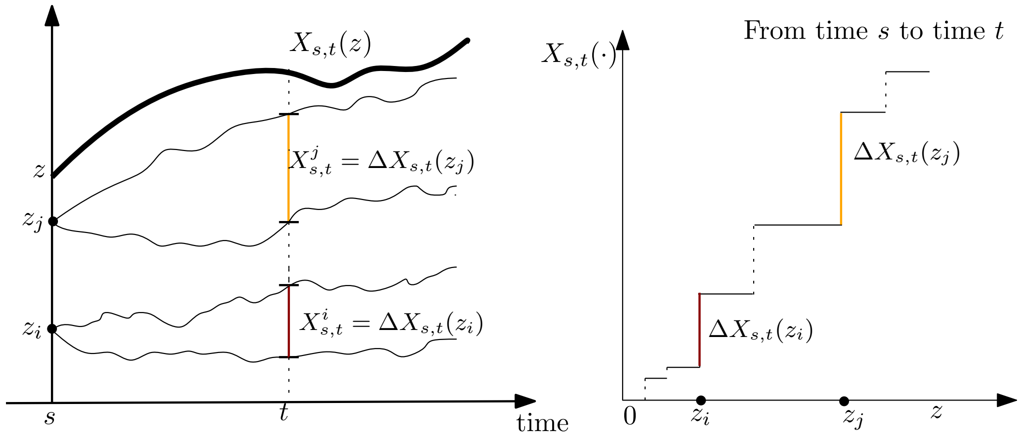


Figure 2.1 – Three-parameter flow

We reexplain here the different scenarios for the flow $(X_{s,t}(z), s \le t, z \ge 0)$ according to the branching mechanism Ψ, see Chapter 1.1. Recall that $X_{s,t}(z)$ has the same law as $X_{t-s}(z)$. Hence we see from Table 1.1 in the previous chapter, that for any $s < t$, the subordinator $X_{s,t}$ is driftless, i.e. $d_r = 0$ for all $r \ge 0$ in (1.1) if and only if $\Psi'(\infty) = \infty$.

As a result, when $\Psi'(\infty) = \infty$, the range $X_{s,t}([0,\infty))$ of the subordinator has zero Lebesgue measure, ensuring that almost every individual z at time t belongs to one of the *infinite* families of ancestors at time s. More precisely, by definition of the genealogy, under the assumption $\Psi'(\infty) = \infty$, almost surely the population at time t, indexed by $\mathbb{R}_+$, can be partitioned according to their ancestor at time s by $\{(X_{s,t}(y-), X_{s,t}(y)], y \in J_{s,t}\}$, where we have set $J_{s,t} := \{x \ge 0 : X_{s,t}(z) \ne X_{s,t}(z-)\}$, the set of jumps of $X_{s,t}$.

When $\Psi'(\infty) < \infty$, there is a positive drift $d_{s,t}$ and to encode the whole population one has to take into account also the continuous increase points of the subordinator $X_{s,t}(\cdot)$. The locations of the latter represent individuals from time s that remain "inactive" until time t. The values of the subordinator at those continuity points are labelling the "inactive" individuals along time.

An important setting which simplifies a lot the interpretation is when Grey's condition $\int^\infty \frac{du}{|\Psi(u)|.} < \infty$ holds. In this case $\Psi'(\infty) = \infty$, so that there is no drift and furthermore for any $t > s$, $\ell_{s,t}((0,\infty] = \ell_{t-s}((0,\infty]) < \infty$, so that the subordinators $X_{s,t}$ are compound Poisson processes. In particular, the set $J_{s,t}$ is the set of atoms of a Poisson process with intensity $u_{t-s}(\infty)$. Note that the partition $\{(X_{s,t}(y-), X_{s,t}(y)], y \in J_{s,t}\}$ consists in a family of *consecutive* intervals. We shall see later how to follow their coalescences in time with the help of a certain coalescent process.

2.3 The inverse flow and the Ancestral Lineage Process

All genealogical relationships are encoded by the flow. In other words, if we consider an individual at a given generation, one can locate its ancestor at any previous generation. The mathematical object that will enable us to do so is the *inverse flow* that we now introduce.

We first define, for $s \le t$ and $x \ge 0$

$$X_{s,t}^{-1}(x) := \inf\{z : X_{s,t}(z) > x\}.$$

The process $X_{s,t}^{-1}$ is the right-continuous inverse of the càdlàg process $X_{s,t}$. Note that the individual $X_{s,t}^{-1}(y)$ is the ancestor alive at time s of the individual y considered at time $t \ge s$. This is the key observation in our study of the genealogy of a CSBP backwards in time. We call inverse flow the process $(\widehat{X}_{s,t}(x), s \le t, x \ge 0)$ defined for all $s \le t, x \ge 0$ as follows

$$\widehat{X}_{s,t}(x) = X_{-t,-s}^{-1}(x). \tag{2.4}$$

We first list some straightforward properties of inverse flows.

Lemma 2.1 (Lemma 3.3 in [F9]). *The following properties hold:*

1. *Almost surely, for every $s \le t$ and $z, x > 0$, we have*
$$\{X_{s,t}(z) > x\} = \{\widehat{X}_{-t,-s}(x) < z\}.$$
2. *For every $t \ge 0$, $(\widehat{X}_{r,s}, r \le s \le t)$ and $(\widehat{X}_{r,s}, t \le r \le s)$ are independent.*
3. *For every $s \le t$, almost surely, for all u, $\widehat{X}_{s,u} = \widehat{X}_{t,u} \circ \widehat{X}_{s,t}$.*
4. *For all $x \ge 0$, $\widehat{X}_{0,0}(x) = x = \lim_{t\to 0} \widehat{X}_{0,t}(x)$ in probability.*

Those results are mostly immediate consequences of properties of the forward flow stated in Definition 2.A and of well-known properties of right-continuous inverses. We skip details here. We shall denote by $(\widehat{X}_t(y), y \ge 0, t \ge 0)$ the flow of inverse subordinators $(\widehat{X}_{0,t}(y), y \ge 0, t \ge 0)$. As noted above, it tracks backwards in time the ancestral lineages of the population at time 0. We therefore also call it the *ancestral lineage process* (ALP for short). Since individuals are ordered, $\widehat{X}_t(y)$ can also be interpreted as the random size of the population at time $-t$ whose descendance at time 0 has size y. Observe that by Lemma 2.1(i) and Definition 2.A(i), we have

$$\forall s \le t,\ \forall z, x \ge 0,\ \mathbb{P}(X_{s,t}(z) > x) = \mathbb{P}(\widehat{X}_{s,t}(x) < z). \tag{2.5}$$

We see here in (2.5) the Siegmund duality relationship i.e. the H-duality with $H(x,y) = \mathbb{1}_{\{x<y\}}$. It is moreover a pathwise relationship since both processes are functional of the flow, see also Lemma 2.1-(i).

Let $\mathbb{e}_q$ be an independent exponential random variable with parameter q, we have

$$\mathbb{P}(\widehat{X}_t(\mathbb{e}_q) > z) = \mathbb{P}(X_{-t,0}(z) < \mathbb{e}_q) = \mathbb{E}[e^{-qX_{-t,0}(z)}] = e^{-zu_t(q)},$$

this simple observation allows us to establish the following first theorem.

Theorem 2.2 (Theorem 3.5 and Proposition 3.6 in [F9]). *Fix $x \in (0,\infty)$. The process $(\widehat{X}_t(x), t \ge 0)$ is a positive Markov process valued in $(0,\infty)$. Its semigroup $(Q_t, t \ge 0)$ satisfies for any bounded measurable function f and any $t \ge 0$*

$$\mathbb{E}[Q_t f(\mathbb{e}_q)] = \mathbb{E}[f(\mathbb{e}_{u_t(q)})] \quad \text{for all } q > 0 \tag{2.6}$$

where $\mathbb{e}_q$ and $\mathbb{e}_{u_t(q)}$ are exponential random variable with parameter q and $u_t(q)$.

The boundaries 0 and ∞ are classified as follows:

1. *The boundary* 0 *is an entrance boundary of* $(\widehat{X}_t, t \geq 0)$ *if and only if* $\int^{\infty} \frac{\mathrm{d}u}{\Psi(u)} < \infty$. *In that case,* $(Q_t, t \geq 0)$ *is extended to* $[0, \infty)$ *by*

$$Q_t f(0) = \int_0^{\infty} f(y) u_t(\infty) e^{-y u_t(\infty)} \mathrm{d}y.$$

   Otherwise, we set $Q_t f(0) = f(0)$.

2. *The boundary* ∞ *is an entrance boundary of* $(\widehat{X}_t, t \geq 0)$ *if and only if* $\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} < \infty$. *In that case,* $(Q_t, t \geq 0)$ *is defined over* $(0, \infty]$ *with*

$$Q_t f(\infty) = \int_0^{\infty} f(y) u_t(0) e^{-y u_t(0)} \mathrm{d}y.$$

   Otherwise, we set $Q_t f(\infty) = f(\infty)$.

Last, the semigroup $(Q_t, t \geq 0)$ *defined over* $[0, \infty]$ *is Feller.*

Observe that (2.6) characterizes the semigroup Q_t, by identification of the Laplace transforms, as it can be rewritten as: for all $q \geq 0$,

$$\int_0^{\infty} q e^{-qx} Q_t f(x) \mathrm{d}x = \int_0^{\infty} u_t(q) e^{-u_t(q)x} f(x) \mathrm{d}x,$$

therefore $Q_t f$ is the inverse Laplace transform of $q \mapsto \frac{u_t(q)}{q} \int_0^{\infty} e^{-u_t(q)x} f(x) \mathrm{d}x$.

Remark 2.3. We know only two cases for which the semigroup Q_t can be made explicit, the Feller diffusion and the Neveu case. In the first case, Q_t is the semigroup of a Feller process with constant immigration; in the latter, Q_t is related to the Mittag-Leffler law.

The above theorem shows that the semigroup of $(\widehat{X}_t)$ can be expressed in simple terms when applied to exponential distributions. This will motivate later on the study of the action of the flow $\widehat{X}$ on Poisson point processes, see Section 2.4.

Remark 2.4. We see here that the duality has exchanged the nature of the boundaries: if the process X gets extinct (which means here that 0 is an exit boundary), then $\widehat{X}$ has its boundary 0 entrance, and similarly for the boundary ∞. This is a general consequence of the duality (not necessarily the Siegmund one) and it will be at the core in the next chapters.

Remark 2.5. The Markov processes $(\widehat{X}_t(0), t \geq 0)$ and $(\widehat{X}_t(\infty), t \geq 0)$ have moreover the following interpretations, in terms of the CSBP:

1. The process $(\widehat{X}_t(0), t \geq 0) = (\widehat{X}_{0,t}(0), t \geq 0)$, starting from 0 at time 0, represents the smallest individual at generation $-t$ to have descendants at time 0. If $\int^{\infty} \frac{\mathrm{d}u}{|\Psi(u)|} < \infty$, there is extinction in finite time for the CSBP X (i.e. with positive probability, $X_{-t,0}(x) = 0$). In that case $(\widehat{X}_{0,t}(0), t \geq 0)$ is a non-trivial Markov process. If $\int^{\infty} \frac{\mathrm{d}u}{|\Psi(u)|} = \infty$, there is no extinction in finite time for the CSBP, thus all individuals at time t have descendants at time 0, $(\widehat{X}_t(0), t \geq 0) \equiv 0$.
2. The process $(\widehat{X}_t(\infty), t \geq 0)$, starting from ∞, represents the smallest individual at generation t with an infinite progeny at time 0. If $\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} < \infty$, there is explosion in finite time for the CSBP X (i.e. with positive probability, $X_{-t,0}(x) = \infty$). In that case, $(\widehat{X}_{0,t}(\infty), t \geq 0)$ is a non-trivial Markov process. If $\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} = \infty$, there is no explosion in finite time and all individuals at time t have finitely many descendants at time 0. Thus $(\widehat{X}_t(\infty), t \geq 0) \equiv \infty$ and $Q_t f(\infty) := f(\infty)$.

2.3.1 Recurrence/transience

By transience, we mean that $\widehat{X}_t(x) \underset{t\to\infty}{\longrightarrow} \infty$ a.s. for any $x \in (0, \infty)$.

Proposition 2.6 (Proposition 3.8 in [F9])**.** *The long-term behaviors of the ALP are as follows:*

1. *if* Ψ *is supercritical, then* $\widehat{X}$ *is positive recurrent with stationary law* $\mathbb{e}_\varrho$*;*
2. *if* Ψ *is subcritical, then* $\widehat{X}$ *is transient;*
3. *if* Ψ *is critical, then* $\widehat{X}$ *is transient if and only if* $\int_0 \frac{u}{\Psi(u)}\mathrm{d}u < \infty$*, otherwise it is null recurrent.*

Example 2.7. If $\Psi(q) = dq^\alpha$ with $1 \le \alpha \le 2$ and $d > 0$ then $(\widehat{X}_t, t \ge 0)$ is null recurrent if $\alpha = 2$ and transient if $\alpha < 2$.

Sketch of proof. The first statement is a direct consequence of the duality relationship (2.5), since $\mathbb{P}(X_t(z) > x)$ as t goes to ∞ converges to the probability of non-extinction, namely $e^{-z\varrho}$. Intuitively, in the supercritical case, the infinite size population model is founded at time $-\infty$ by a single individual located on $[0, \infty)$ at a point $\mathbb{e}_\varrho$, distributed as an exponential random variable with parameter ϱ. In other words, all individuals below $\mathbb{e}_\varrho$ have vanishing descendants and all above have common ancestor $\mathbb{e}_\varrho$.

To explain the condition $\int_0 \frac{x}{\Psi(x)}\mathrm{d}x = \infty$ in the critical case. We mention that the latter is known to entail that the first passage times below any positive level of the CSBP have infinite mean, see [F6], in such case the forward process takes a lot of time to go below any level although it will ultimately do. Symmetrically the process $(\widehat{X}_t, t \ge 0)$ will make large oscillations and is null recurrent.

In the subcritical case, for any fixed $a > 0$, individuals below level a living at arbitrarily large times in the past will, heuristically, have no progeny at time 0. Therefore, the ancestral lineage of an individual x living at time 0 will go above any fixed level a as time goes to ∞. This explains the transience. More precisely, Siegmund duality allows one to show that the potential measure of $\widehat{X}$ gives finite mass to any interval of the form $(0, a)$ with $a > 0$ if and only if $\int_0 \frac{x}{\Psi(x)}\mathrm{d}x < \infty$. A classical result from general theory of Markov processes, see e.g. Proposition 2.2-(iv') in Getoor [65], entails that the last exit time from $(0, a)$ is finite. Since this is true for all a, it follows that the process $(\widehat{X}_t, t \ge 0)$ is transient. □

2.3.2 Feller flow (Section 4 of [F9])

The simplest case of inverse flow is in the setting of a CSBP X that is a Feller diffusion, say with branching mechanism $\Psi(q) = \frac{\sigma^2}{2}q^2 + \beta q$ with $\beta \in \mathbb{R}$. In this case, one can check by hands[4] that the inverse flow is a flow of continuous-state branching processes with immigration with mechanisms $\widehat{\Psi}(q) = \frac{\sigma^2}{2}q^2 + \beta q$ and linear immigration $\widehat{\Phi}(q) := \frac{\sigma^2}{2}q$. That is to say, for any fixed $x \ge 0$, $(\widehat{X}_t(x), t \ge 0)$ is a Markov process with semigroup given by

$$\mathbb{E}[e^{-z\widehat{X}_t(x)}] = e^{-x\widehat{u}_t(z) - \frac{\sigma^2}{2}\int_0^t \widehat{u}_s(z)\mathrm{d}s},$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{u}_t(z) = -\widehat{\Psi}(\widehat{u}_t(z)), \widehat{u}_0(z) = z.$$

4. The key fact is that the jump distribution of the compound Poisson process $X_t(\cdot)$ is the exponential law, by inverting they become the inter-arrival times of $\widehat{X}_t(\cdot)$ whose jumps are also exponentially distributed. All parameters can be computed explicitly.

In particular, $(\widehat{X}_t(x) - \widehat{X}_t(0), t \geq 0, x \geq 0)$ is a flow of Feller CSBPs with branching mechanism $\widehat{\Psi}$ and as explained in Chapter 1, it can be represented as follows: for all $t > 0$,

$$\widehat{X}_t(x) = \widehat{X}_t(0) + \sum_{\substack{x_i \leq x \\ i \in I}} \widehat{X}^i_t$$

where $\mathcal{N} = \sum_{i\in I} \delta_{(x_i, \widehat{X}^i)}$ is a PPP with intensity $\mathrm{d}x \otimes \mathbb{N}_{\widehat{\Psi}}(\mathrm{d}X)$ and $\mathbb{N}_{\widehat{\Psi}}$ is the cluster measure. The atoms $(\widehat{X}^i, i \in I)$ can be interpreted as the ancestral lineages of the initial individuals $(x_i, i \in I)$. They are intuitively independent Feller diffusions with mechanism $\widehat{\Psi}$ starting from infinitesimal masses. For any $i \in I$, denote by $\zeta_i := \inf\{t \geq 0; \widehat{X}^i_t = 0\}$. The time ζ_i represents a binary coalescence time between two "consecutive" individuals. By definition of $\mathbb{N}_{\widehat{\Psi}}$, for any $t > 0$, $\mathbb{N}_{\widehat{\Psi}}(\zeta > t) = \widehat{u}_t(\infty)$. The PPP, $\sum_{i\in I} \delta_{(x_i,\zeta_i)}$, coincides with the Coalescent Point Process of Aldous and Popovic [2], see also Popovic [125] and the Comb representation of Lambert and Uribe-Bravo [95]. We represent the ancestral lineages in the Feller case and their coalescences in Figure 2.2. Recall also from Remark 2.5 that $\widehat{X}_t(0)$ is the first individual from generation $-t$ to have descendants at time 0.

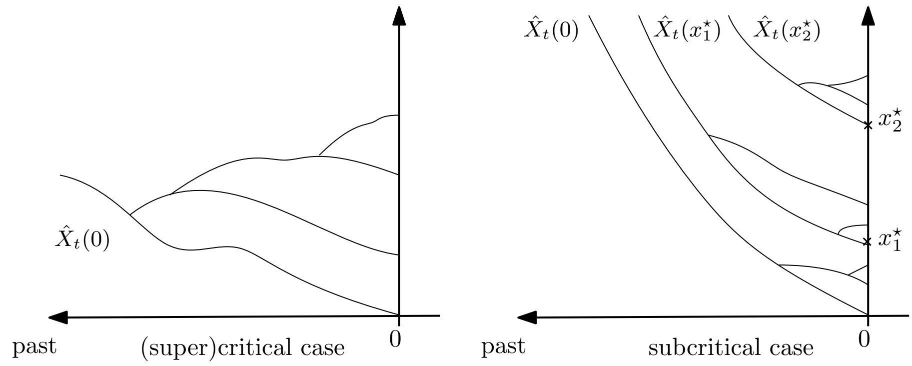


Figure 2.2 – Schematic representation of ancestral lineages and their binary coalescences

In the subcritical case, $(\beta < 0)$, $(\widehat{X}_t(x) - \widehat{X}_t(0), x \geq 0, t \geq 0)$ is a flow of *supercritical* CSBPs. Recall the notion of prolific individuals introduced in Chapter 1, see Definition 1.2 and denote them by $(x^\star_n, n \geq 1)$. By definition,

$$x^\star_1 := \inf\{x \geq 0; \widehat{X}_t(x) - \widehat{X}_t(0) \underset{t\to\infty}{\longrightarrow} \infty\} \text{ and } x^\star_{n+1} := \inf\{x \geq x^\star_n; \widehat{X}_t(x) - \widehat{X}_t(0) \underset{t\to\infty}{\longrightarrow} \infty\}$$

and they form the jumps times [5] of a Poisson process with intensity $\widehat{\varrho} := -\frac{2\beta}{\sigma^2}$. Within the framework of inverse flow, the random partition of $\mathbb{R}_+$: $([0, x^\star_1), [x^\star_1, x^\star_2), ...)$ corresponds to current families with distinct common ancestors.

When the branching mechanism Ψ is not of the quadratic form, multiple births occur in the population. Thus, when time runs backward, coalescences of multiple lineages arise. The law of the inverse flow $\widehat{X}$ becomes then much more involved. The story told however above for the Feller flow can be generalized in several directions. This will be the aim of the next sections. Before doing so, we state here a martingale problem solved by the inverse flow.

5. times in the x-axis

2.3.3 Martingale problem

We give here the extended generator $\widehat{\mathscr{L}}$ of $(\widehat{X}_t, t \geq 0)$. Recall $\mathscr{L}^\Psi$ the generator of the CSBP with mechanism Ψ.

As we consider the flow of subordinators over $[0, \infty]$, it is natural to express $\mathscr{L}^\Psi$, given in (A.18), as follows. For all function G in C_b^2 the space of twice differentiable bounded functions with bounded derivatives, we have

$$\mathscr{L}^\Psi G(x) = \frac{\sigma^2}{2} x G''(x) + \beta x G'(x) + \int_0^\infty \pi(\mathrm{d}h) \int_0^\infty \mathrm{d}u \left(G(\Delta_{h,u}(x)) - G(x) - h \mathbb{1}_{\{u \leq x\}} G'(x) \mathbb{1}_{\{h \leq 1\}} \right) \tag{2.7}$$

with $\Delta_{h,u}(x) := x + h\mathbb{1}_{\{x \geq u\}}$.

Heuristically, $\Delta_{h,u}$, see Figure 2.3, represents the reproduction of the individual u into a mass h of children in an infinitesimal time. Set $\psi_{h,u} := \Delta_{h,u}^{-1}$ the right-continuous inverse function of $\Delta_{h,u}$, see Figure 2.3.

$$\psi_{h,u}(z) = z\mathbb{1}_{[0,u]}(z) + u\mathbb{1}_{[u,u+h]}(z) + (z-h)\mathbb{1}_{[u+h,\infty)}(z).$$

For any $y \geq 0$, if individual u has at time t a progeny of size h, then $\psi_{h,u}(y)$ at time $t-$ is the infinitesimal parent of individual y at time t: if $y < u$, then y has no parent but himself, if $y \in [u, u+h]$, the parent of y is $\psi_{h,u}(y) = u$, if $y > u + h$ then its parent is $\psi_{h,u}(y) = y - h$. If $y_1 \neq y_2$ then $\psi_{h,u}(y_1) = \psi_{h,u}(y_2)$ if and only if $y_1, y_2 \in [u, u+h]$.

Intuitively, the forward flow of subordinators, see Definition 2.A, is obtained by composing the elementary functions $\Delta_{h,u}$. This is represented at the top of the image below. Taking the inverse flow leads to compose functions $\psi_{h,u}$, this is represented at the bottom.

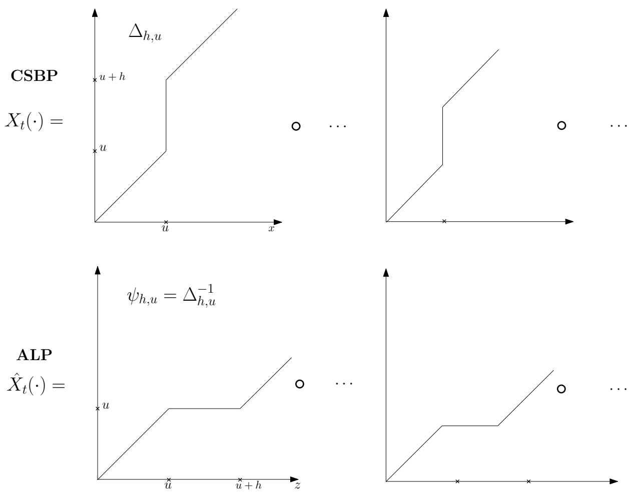


Figure 2.3 – Composition of functions $\Delta_{h,u}$ and $\psi_{h,u}$

Theorem 2.8 (Theorem 6.1 in [F9]). *For any function F in C_0^2, set*

$$\widehat{\mathscr{L}} F(z) = \frac{\sigma^2}{2} z F''(z) + \left(\frac{\sigma^2}{2} - \beta z \right) F'(z) + \int_0^\infty \pi(\mathrm{d}h) \int_0^\infty \mathrm{d}u \left[F(\psi_{h,u}(z)) - F(z) + h\mathbb{1}_{\{h \leq 1\}} F'(z) \mathbb{1}_{\{z > u\}} \right].$$

Then for any $y > 0$, $(\widehat{X}_t(y), t \geq 0)$ *solves the following well-posed martingale problem*

$$\text{(MP)} \qquad \left(F(\widehat{X}_t(y)) - \int_0^t \widehat{\mathscr{L}}F(\widehat{X}_s(y))\mathrm{d}s, t \geq 0\right)$$

is a martingale for any function F *in*

$$\mathscr{D} := \left\{F \in C_0^2; F' \in L^1 \text{ and } \beta x F'(x), \frac{\sigma^2}{2} x F''(x) \underset{x\to\infty}{\longrightarrow} 0\right\}.$$

The expression given above provides the dynamics of the inverse flow but is a bit hard to read if we look for the position of the process right after the jump from a given z. We rewrite it, then, in its classical Von-Wandelfelds-Courrège form. For any $f \in C_b^2$,

$$\widehat{\mathscr{L}}f(z) = \frac{\sigma^2}{2} z f''(x) + \int_0^z \left[f(z-h) - f(z) + hf'(z)\right] \nu(z, \mathrm{d}h) + b(z)f'(z) \tag{2.8}$$

with

$$\nu(z, \mathrm{d}h) := \mathbb{1}_{\{h\leq z\}}((z-h)\pi(\mathrm{d}h) + \overline{\pi}(h)\mathrm{d}h)$$

and

$$b(z) := \int_0^\infty h(z\mathbb{1}_{\{h\leq 1\}}\pi(\mathrm{d}h) - \nu(z, \mathrm{d}h)) - \beta z + \frac{\sigma^2}{2}.$$

Notice that it does not take a particularly nice form in general, apart from the Feller case (for which $\pi \equiv 0$) and the Neveu case (for which $\pi(\mathrm{d}h) = \frac{\mathrm{d}h}{h^2}$). In the latter case, we recover a process (called Mittag-Leffler process in [115]) that was introduced by Möhle in [115], Kukla and Möhle in [87] and Baur and Bertoin [10].

In the next section, we seek a dynamical description of the coalescences. We will construct a simple class of non-exchangeable Markovian coalescents which will allow us to encode easily multiple coalescences in lineages backwards in time. This will be in the same spirit as Bertoin and Le Gall's flow of bridges, [20], in which it is established that the Λ-coalescents (we will present them briefly in Part III) are encoding the genealogy of Fleming-Viot processes.

The ancestral lineage process $(\widehat{X}_t(x), t \geq 0)$ for a fixed individual $x \geq 0$ is studied further in Section 2.5.

2.4 Consecutive coalescents

2.4.1 Consecutive coalescents in CSBPs through Poisson sampling

To follow the coalescences, we are going to use the coagulation operator. The partitions involved have blocks of the shape of integer intervals [6]. We index them by their least element: $C = (C_1, C_2, \cdots)$ and the first element in C_i is $1 + \#C_{i-1}$. We call them consecutive partitions. We denote by $0_{\mathbb{N}}$ and $1_{\mathbb{N}}$ respectively the partition of $\mathbb{N}$ into singletons and into a single block $\{\{\mathbb{N}\}, \emptyset, \cdots\}$.

Definition 4.A (Coagulation operator, see Chapter 4.2 in [18])**.** *For any partitions* C, C' *with* n *and* n' *blocks such that* $\#C \leq n'$, *we define the partition* $\mathrm{Coag}(C, C')$ *by*

$$\mathrm{Coag}(C, C')_j = \textstyle\bigcup_{i\in C'_j} C_i \quad \text{for any } j \in \mathbb{N}.$$

6. the Coag operator still simplifies the discussion even with so simple partitions

When C and C' are consecutive partitions, as each block of $\text{Coag}(C, C')$ is the union of a consecutive sequence of consecutive blocks, obviously $\text{Coag}(C, C')$ is again a consecutive partition.

We now construct consecutive coalescent processes related to the genealogy of the flow of subordinators $(X_{s,t}(z), s \leq t, x \geq 0)$. Denote by $(J_i^\lambda, i \geq 1)$ the sequence of atoms of an independent Poisson process with intensity λ. For any $t \geq 0$, we define $C^\lambda(t)$ as

$$i \overset{C^\lambda(s,t)}{\sim} j \text{ if and only if } \widehat{X}_{s,t}(J_i^\lambda) = \widehat{X}_{s,t}(J_j^\lambda). \tag{2.9}$$

Remark 2.9. There is a very strong parallel between the consecutive coalescents as defined above and the exchangeable ones. Loosely speaking, if one replaces the subordinators and the Poisson arrival "times", J_i^λ's respectively by exchangeable bridges and uniform random variables on $[0, 1]$, we would get exchangeable coalescents, see [20] and [18, Chapter 4.4].

The next theorem describes the law of the process $(C^\lambda(t), t \geq 0) := (C^\lambda(0, t), t \geq 0)$.

Theorem 2.10 (Theorem 5.10 in [F9])**.** *For any $\lambda > 0$, the partition-valued process $(C^\lambda(t), t \geq 0)$ is a consecutive coalescent, in the sense that for any $s, t \geq 0$,*

$$C^\lambda(t + s) \overset{Law}{=} \text{Coag}(C^\lambda(t), C')$$

for some independent random consecutive partition C' with law depending on s and t. Furthermore, $C^\lambda(0) = 0_{\mathbb{N}}$ and

— *The lengths of the blocks of $C^\lambda(t)$ are i.i.d. with law characterized by*

$$\mathbb{E}[z^{\#C_1^\lambda(t)}] = 1 - \frac{u_t(\lambda(1 - z))}{u_t(\lambda)} \text{ for any } z \in [0, 1]. \tag{2.10}$$

— *There are no simultaneous coalescences and for any $k \geq 2$, the rate at time t at which k given <u>consecutive</u> blocks of $C^\lambda(t-)$ merge is*

$$\mu_t^\lambda(k) := \frac{\sigma^2}{2} u_t(\lambda) 1_{\{k=2\}} + u_t(\lambda)^{k-1} \int_{(0,\infty)} \frac{\lambda^k}{k!} e^{-u_t(x)x} \pi(\mathrm{d}x). \tag{2.11}$$

Notice that since C^λ starts from infinitely many blocks, the first coalescence event occurs immediately. To give a rigorous sense to the coalescence rate in (2.11), one should first look at the restriction over finite sets and then argue by compatibility as in the usual theory of coagulation/fragmentation.

For the sake of clarity, we explain it further below. Fix a time t. For any $n \geq 1$, denote by $C^\lambda_{|[n]}(t)$, the partition of $[n] := \{1, \cdots, n\}$, given by $(C_i^\lambda(t) \cap [n], i \geq 1)$. Conditionally given $\#C_{|[n]}(t-) = m$, consider for any $j \leq m - 1$, the consecutive partitions of $[m]$:

$$C_{\text{in}}^{j,k} := (\{1\}, ..., \{j, ..., j + k - 1\}, ..., \{m\}) \text{ for any } 2 \leq k \leq m - j, \text{ and}$$
$$C_{\text{out}}^{j} := (\{1\}, ..., \{j, ..., m\}).$$

Attach to each $C_{\text{in}}^{j,k}$ and C_{out}^{j} respectively a random clock $\zeta_{\text{in}}^{j,k}$ and ζ_{out}^{j} with law

$$\mathbb{P}(\zeta_{\text{in}}^{j,k} > s) = \exp\left(-\int_0^s \mu_r^\lambda(k)\mathrm{d}r\right) \text{ and } \mathbb{P}(\zeta_{\text{out}}^{j} > s) = \exp\left(-\int_0^s \overline{\mu}_r^\lambda(m - j + 1)\mathrm{d}r\right).$$

Then the process jumps from the partition $C_{|[n]}(t-)$ to $\text{Coag}(C_{|[n]}(t-), D)$ with D the partition in $\{C_{\text{in}}^{j,k}, C_{\text{out}}^{j}\}$ associated with the first random clock that rings.

The subscripts "in" and "out" underline the fact that the coalescence either involve only blocks inside $C^{\lambda}_{[n]}$ or at the contrary is part of a coalescence event in the partition restricted to a bigger set.

Last, the compatibility by restriction, i.e. the fact that for any $n' \geq n$, $C^{\lambda}_{|[n]}(\cdot) = C^{\lambda}_{|[n']}(\cdot)_{|[n]}$ is easily checked and ensures that the family $(\mu^{\lambda}_t(\cdot), t > 0)$ characterizes the law of C^{λ}.

We gather in the next theorem basic properties of $(C^{\lambda}(t), t \geq 0)$.

Theorem 2.11 (Propositions 5.18 and 5.19 in [F9])**.** *Fix $\lambda > 0$.*

- *If Ψ is critical or supercritical then $(C^{\lambda}(t), t \geq 0)$ converges almost-surely towards the partition with a single block $1_{\mathbb{N}}$.*
- *If Ψ is subcritical, then the process $(C^{\lambda}(t), t \geq 0)$ converges almost-surely towards a partition $C^{\lambda}(\infty)$, whose law is characterized by*

$$\mathbb{E}[z^{\#C^{\lambda}_1(\infty)}] = 1 - e^{-\Psi'(0^+)\int_{\lambda(1-z)}^{\lambda} \frac{du}{\Psi(u)}} \text{ for any } z \in (0,1).$$

  In this case, the individuals $(J^{\lambda}_1, J^{\lambda}_2, \ldots)$ belong to different families with i.i.d sizes distributed as $\#C^{\lambda}_1(\infty)$.

Furthermore, for any $t \geq 0$, there are infinitely many singleton blocks at time t and

$$\frac{\#\{i \in [n]; \#C^{\lambda}_i(t) = 1\}}{n} \underset{n\to\infty}{\longrightarrow} D^{\lambda}_t := \frac{\lambda}{\Psi(\lambda)} \frac{\Psi(u_t(\lambda))}{u_t(\lambda)} \text{ a.s.},$$

and D^{λ}_t is the proportion of sampled individuals whose lineages have not been involved in coalescences by time t.

2.4.2 Backward genealogy of the whole population

In the previous section, we have defined some coalescent processes arising from sampling initial individuals along a Poisson process with an arbitrary intensity λ. The consecutive coalescents obtained by this procedure are only approximating the backward genealogy. They give the genealogy of a random sample of the population. The objective of this subsection is to observe that when the Grey's condition holds, one can define a consecutive coalescent matching with the complete genealogy of the population from any positive time. In all this section, assume the Grey's condition [7]

$$\int^{\infty} \frac{\mathrm{d}x}{|\Psi(x)|} < \infty. \tag{2.12}$$

Heuristically, we make $\lambda \to \infty$ in Theorem 2.10, to study the genealogy of the whole population. The limiting process would indeed characterize the genealogy of the CSBP as in this case, an everywhere dense sub-population would be sampled and its genealogy given, which is enough to deduce the genealogical relationship between any pair of individuals. However, this method cannot work directly as one would have jump rates that may explode.

7. Recall that this excludes processes with finite variation, and that any Ψ verifying (2.12) is positive near ∞, so that (2.12) is equivalent to $\Psi'(\infty) = \infty$ and $\int^{\infty} \frac{du}{\Psi(u)} < \infty$.

Sampling all individuals from time $s > 0$

Fix a time $s > 0$. Recall from Table 1.1, that the subordinator $(X_{-s,0}(x), x \geq 0)$ is a compound Poisson process with Lévy measure $\ell_s(\mathrm{d}x)$ independent of $(X_{-t,-s}(x), x \geq 0, t \geq s)$. Let $(J_i^{u_s(\infty)}, i \geq 1)$ be the jump times of $(X_{-s,0}(x), x \geq 0)$. They are atoms of a Poisson process with intensity $u_s(\infty) = \ell_s([0,\infty])$, independent of $(\widehat{X}_{s,t}, t \geq s)$. Consider $(C(s,t), t \geq s)$ the partition-valued process defined by

$$i \overset{C(s,t)}{\sim} j \text{ iff } \widehat{X}_{s,t}(J_i^{u_s(\infty)}) = \widehat{X}_{s,t}(J_j^{u_s(\infty)}).$$

The process $(C(s,t), t > s)$ provides a dynamical description of the genealogy of initial individuals whose most recent common ancestors are found at time $s > 0$.

The following theorem is a direct application of Theorem 2.10.

Theorem 2.12 (Theorem 5.21 in [F9])**.** *For any $s > 0$, the partition-valued process $(C(s,t), t \geq s)$ is a consecutive coalescent started from $0_{\mathbb{N}}$.*

— *For any $t \geq s$, the lengths of the blocks are i.i.d. with law characterized by*

$$\mathbb{E}[z^{\#C_1(s,t)}] = 1 - \frac{u_{t-s}(u_s(\infty)(1-z))}{u_t(\infty)} \text{ for any } z \in (0,1).$$

— *Any $k \geq 2$ consecutive blocks of $C(s,t-)$ merge at rate*

$$\mu_t^{\infty}(k) := \frac{\sigma^2}{2} u_t(\infty) 1_{\{k=2\}} + u_t(\infty)^{k-1} \int_{(0,\infty)} \frac{x^k}{k!} e^{-u_t(\infty)x} \pi(\mathrm{d}x). \tag{2.13}$$

The link with the reduced trees and the prolific tree (Corollary 5.22 in [F9])

With no surprise, we recover by reversing time in the consecutive coalescent the *reduced* trees. Fix an horizon time $T > 0$ and consider the consecutive partitions $C(T-t, T)$ for any $t \in [0, T[$. The processes $(Z_i^T(t), 0 \leq t < T) := (\#C_i(T-t, T), 0 \leq t < T)$ are i.i.d inhomogeneous continuous-time Galton-Watson processes. For any $z \in [0,1]$, and any $t \in [0, T[$

$$\mathbb{E}[z^{Z_i^T(t)}] = 1 - \frac{u_t(u_{T-t}(\infty)(1-z))}{u_T(\infty)}. \tag{2.14}$$

Moreover, denoting by γ_i^T, the time of its first jump, one has for any $t \in [0, T[$

$$\mathbb{P}(\gamma_i^T > t) = \frac{\Psi(u_T(\infty))}{u_T(\infty)} \frac{u_{T-t}(\infty)}{\Psi(u_{T-t}(\infty))}.$$

We refer to Duquesne and Le Gall [47, Theorem 2.7.1] for the (sub)critical case, see also Fekete et al. [58] for an approach with stochastic differential equations. In the supercritical case, since for any $t \geq 0$, $u_{T-t}(\infty) \underset{T\to\infty}{\longrightarrow} \varrho$, we see in (2.14) that $(Z_1^T(t), t \geq 0)$ converges, as T goes to infinity, in the finite-dimensional sense, towards a Markov process $(Z^{\infty}(t), t \geq 0)$ whose semigroup satisfies for any $z \in (0,1)$

$$\mathbb{E}[z^{Z^{\infty}(t)}] = 1 - \frac{u_t(\varrho(1-z))}{\varrho}.$$

The process $(Z^{\infty}(t), t \geq 0)$ is a continuous-time Galton-Watson process, homogeneous in time, with reproduction measure μ^{ϱ}, see (2.11), which, using the fact that $u_t(\varrho) = \varrho$, takes the form

$$\mu^{\varrho}(k) := \frac{\sigma^2}{2} \varrho 1_{\{k=2\}} + \varrho^{k-1} \int_{(0,\infty)} \frac{x^k}{k!} e^{-\varrho x} \pi(\mathrm{d}x).$$

This is nothing but the *prolific* tree mentioned at the end of Chapter 1.2.

Genealogy of the whole population

The coalescent process $(C(s,t), t \geq s)$ only describes coalescence in families from time $s > 0$. Our objective here is to define a coalescent process from time 0. Denote by $\mathscr{C}_{\mathbb{R}_+}$ the space of partitions of $(0,\infty)$ into consecutive half-closed intervals. That is to say, partitions of the form $\mathscr{C} = ((0,x_1],(x_1,x_2],...)$ for some non-decreasing sequence of positive real numbers $(x_i, i \geq 1)$.

The space of consecutive partitions of $\mathbb{N}$ endowed with the coagulation operator acts as follows on $\mathscr{C}_{\mathbb{R}_+}$: for any $\mathscr{C} \in \mathscr{C}_{\mathbb{R}_+}$ and $C \in \mathcal{C}_\infty$, for any $i \geq 1$

$$\mathrm{Coag}(\mathscr{C}, C)_i = \bigcup_{j \in C_i} \mathscr{C}_j$$

where $\mathscr{C}_j = (x_{j-1}, x_j]$ and $x_0 = 0$.

The following theorem achieves one of our goals by describing completely the genealogy backwards in time as well as the sizes of asymptotic families. For any $t > 0$, denote by J_{-t} the set of jumps of the subordinator $(X_{-t,0}(x), x \geq 0)$.

The quasi-stationary distribution of the CSBP (forward in time), denoted by ν_∞, see Theorem A.24 in Annex A, comes here into play.

Theorem 2.13 (Theorem 5.24 in [F9])**.** *Define the process $(\mathscr{C}(t), t > 0)$ valued in $\mathcal{C}_{\mathbb{R}+}$ as follows:*

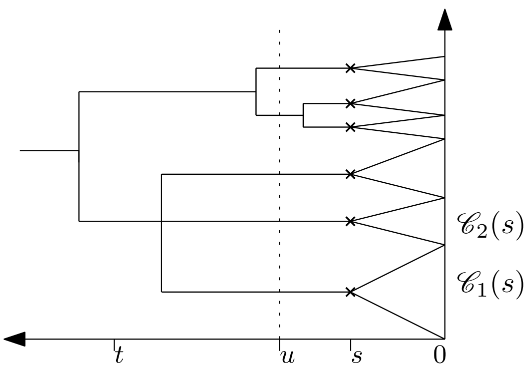
$$\mathscr{C}(t) = \{(X_{-t,0}(x-), X_{-t,0}(x)], x \in J_{-t}\}.$$

The process $(\mathscr{C}(t), t > 0)$ is a time-inhomogeneous Markov process such that for any $t \geq s > 0$,

$$\mathscr{C}(t) = \mathrm{Coag}(\mathscr{C}(s), C(s,t)) \ a.s.$$

- *In the critical or supercritical case, $\mathscr{C}(t) \underset{t\to\infty}{\longrightarrow} \mathbb{1}_{(0,\infty)}$ a.s.*
- *In the subcritical case, $\mathscr{C}(t) \underset{t\to\infty}{\longrightarrow} \mathscr{C}(\infty)$ a.s. and the length of a typical interval at the limit has for law the quasi-stationary distribution ν_∞,*

$$\mathbb{E}[e^{-u|\mathscr{C}_1(\infty)|}] = \int_0^\infty e^{-uy}\nu_\infty(\mathrm{d}y) = 1 - \exp\left(-\Psi'(0+)\int_u^\infty \frac{\mathrm{d}v}{\Psi(v)}\right).$$



Intervals at time s are given by $(\mathscr{C}_i(s),\ i \in [6])$

$$C(s,u)_{|[6]} = (\{1\},\{2\},\{3\},\{4,5\},\{6\})$$
$$C(u,t)_{|[5]} = (\{1,2,3\},\{4,5\})$$
$$C(s,t)_{|[6]} = \mathrm{Coag}(C(s,u),C(u,t))_{|[6]} = (\{1,2,3\},\{4,5,6\})$$

Figure 2.4 – Schematic representation of the genealogy

In the figure above, we fix a time $s > 0$, and follow six distinct families represented by the random intervals $(\mathscr{C}_i(s), 1 \leq i \leq 6)$. At time $t > s$, the partition becomes $\mathscr{C}(t) = (\cup_{i=1}^3 \mathscr{C}_i(s), \cup_{i=4}^6 \mathscr{C}_i(s))$.

Under Grey's condition, in the critical and supercritical case, the current population comes from a single ancestor. In the subcritical case however, there are families descending from distinct common ancestors. The families have furthermore i.i.d sizes distributed as the quasi-stationary distribution ν_∞.

So far, the genealogy of the total population has only been characterized under Grey's condition. The latter ensures somehow a discrete structure hidden in the continuous population model. When this condition is not fulfilled the process $(\mathscr{C}(t), t \geq 0)$ cannot be described by a single consecutive coalescent on $\mathbb{N}$. Information about the ancestral partition of randomly sampled individuals are however available through Theorem 2.10. A natural question is to see how the common ancestors from time $-\infty$ are distributed along the half-line when Grey's condition does not hold (in this case no quasi-stationary distribution exist). This will be addressed in the next section via a different method.

2.5 Asymptotics of Ancestral Lineage Processes in the subcritical case

We leave now for good the study of the consecutive coalescent and go back to the ancestral lineage process (ALP) $\widehat{X}$. In the remainder of the chapter, we focus on the subcritical case, $\Psi'(0+) > 0$. As discussed earler in the Chapter, see Proposition 2.6, in this setting, $\widehat{X}$ goes to ∞ a.s.. In the same spirit as Chapter 1, where we revisited the almost sure renormalisations of supercritical CSBPs with finite mean, we will now study the "speed of escape" of the ancestral lineage processes. More precisely, we will see how to renormalize almost surely the flow $(\widehat{X}_t(x), t \geq 0, x \geq 0)$. The interpretation of the limiting process in terms of genealogy is moreover provided and will enable us to answer the question raised at the end of the previous section.

2.5.1 Almost sure renormalisation and ancestral partition ([F15])

Recall the cumulant $t \to u_t$ of the CSBP, see (0.2), the function u is solution to $\frac{\mathrm{d}}{\mathrm{d}t} u_t(z) = -\Psi(u_t(z))$, with $u_0(z) = z$.

Theorem 2.14 (Theorem 3.1 in [F15]). *Assume $\Psi'(0+) > 0$. Fix $z \in (0, \infty)$. Then, almost surely*

$$u_t(z)\widehat{X}_t(x) \underset{t\to\infty}{\longrightarrow} \widehat{W}^z(x), \text{ for all } x \notin J^z := \{x > 0 : \widehat{W}^z(x) > \widehat{W}^z(x-)\},$$

where the process $\widehat{W}^z$ has càdlàg paths and its right-inverse process W^z, defined for any $y \geq 0$ by

$$W^z(y) := \inf\{x \geq 0 : \widehat{W}^z(x) > y\},$$

is a ***driftless*** *subordinator with Laplace exponent κ_z defined by*

$$\kappa_z : \theta \mapsto e^{-\Psi'(0+)\int_\theta^z \frac{\mathrm{d}u}{\Psi(u)}}. \tag{2.15}$$

Remark 2.15. Notice that a change in z affects $\widehat{W}^z$ only by a multiplicative deterministic constant, see (2.15). This stems to the fact that for any subcritical mechanism, one has

$$\frac{u_t(z)}{u_t(z')} \underset{t\to\infty}{\longrightarrow} c_{z,z'} := \exp\left(\Psi'(0+)\int_{z'}^z \frac{\mathrm{d}u}{\Psi(u)}\right) \in (0, \infty). \tag{2.16}$$

Recall π the Lévy measure of the branching mechanism Ψ. The following corollary shows that the ancestral lineage process $(\widehat{X}_t(x), t \geq 0)$ has an exponential growth when the measure π satisfies an $L \log L$ condition.

Proposition 2.16 (Corollary 3.3 in [F15]). *For any $z > 0$, $u_t(z) \underset{t\to\infty}{\sim} c_z e^{-\Psi'(0+)t}$ for some constant $c_z > 0$ if and only if $\int_1^\infty u \log u \pi(\mathrm{d}u) < \infty$. Moreover, under this latter condition, almost surely*

$$e^{-\Psi'(0+)t}\widehat{X}_t(x) \underset{t\to\infty}{\longrightarrow} \widehat{W}(x), \text{ for all } x \notin J := \{x > 0 : \widehat{W}(x) > \widehat{W}(x-)\},$$

where $\widehat{W}$ is the inverse of a subordinator W with Laplace exponent

$$\kappa : \theta \in [0, \infty) \mapsto \theta e^{-\Psi'(0+)\int_0^\theta \left(\frac{1}{\Psi'(0+)u} - \frac{1}{\Psi(u)}\right)\mathrm{d}u}.$$

Remark 2.17. Notice the "symmetry" between Theorem 5.4 and Theorem 2.A:

$$u_t(z)\widehat{X}_t(x) \underset{t\to\infty}{\longrightarrow} \widehat{W}^z(x) \text{ and } u_{-t}(x)X_t(z) \underset{t\to\infty}{\longrightarrow} W^x(z)$$

where we recall that $v_t := u_{-t}$ stands for the inverse of $u_t(\cdot)$, i.e.:

$$\{u_t(z) > x\} = \{z > v_t(x)\}$$

that is to say $t \mapsto v_t$ is the Siegmund dual of $t \mapsto u_t$ in the deterministic setting.

We recover also the $L \log L$ condition for having exactly an exponential growth in both cases, compare Proposition 2.16 with Proposition 2.A.

Example 2.18 (Example 3.5 in [F15]). Let $\gamma > 0$. Consider the subcritical Neveu CSBP whose branching mechanism is defined by $\Psi(u) := \gamma(u+1)\log(u+1)$ for all $u \geq 0$. Note that $\Psi'(0+) = \gamma > 0$ and $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} = \infty$. In this case

$$\kappa(\theta) = \gamma \log(1+\theta) = \int_0^\infty (1 - e^{-\theta x})\gamma \frac{e^{-x}}{x}\mathrm{d}x$$

the limiting process $\widehat{W}$ is therefore an inverse Gamma subordinator, hence has continuous paths and by Proposition 2.16, almost surely

$$e^{-\gamma t}\widehat{X}_t(x) \underset{t\to\infty}{\longrightarrow} \widehat{W}(x) \text{ for all } x \geq 0.$$

We now interpret the process $(\widehat{W}^z(x), x \geq 0)$ in terms of the population model. Define a random equivalence relation $\mathscr{A}$ on $(0, \infty)$ via

$$x \overset{\mathscr{A}}{\sim} y \text{ if and only if } \widehat{W}^z(x) = \widehat{W}^z(y).$$

This induces a random partition [8] of the set $(0, \infty)$ into open intervals of constancy of $\widehat{W}^z$. A direct consequence of (2.16) is that the partition $\mathscr{A}$ does not depend on the parameter z.

By definition, the subintervals of the partition $\mathscr{A}$ are made of individuals whose ancestral lineages have the same asymptotic behaviour. These subintervals correspond to the jump intervals of W^z, the subordinator obtained as the right-inverse of $\widehat{W}^z$, that is to say

$$\mathscr{A} = \{(W^z(x-), W^z(x)); x > 0\} \text{ a.s.}$$

8. up to a Lebesgue negligible set

In other words the families in $\mathscr{A}$ are separated by the elements of the support of the Stieltjes measure $\mathrm{d}\widehat{W}^z$, that is $\mathscr{S} := \overline{\{W^z(x) : x \geq 0\}}$ and their sizes are governed by the Lévy measure of W^z.

The next theorem states that $\mathscr{A}$ corresponds actually to the families of current individuals having a common ancestor and gives the fractal dimension of the ancestors set.

Theorem 2.19 (Theorem 3.7 in [F15], Ancestral partition)**.** *For any $x, y \in (0, \infty)$,*

$$x \overset{\mathscr{A}}{\sim} y \text{ if and only if } \widehat{X}_t(x) = \widehat{X}_t(y) \text{ for some } t \geq 0. \tag{2.17}$$

Moreover, the Hausdorff dimension of $\mathscr{S}$ is

$$\dim_H(\mathscr{S}) = \frac{\Psi'(0+)}{\Psi'(\infty)} \in [0,1) \text{ a.s.} \tag{2.18}$$

Sketch of proof. Clearly if $\widehat{X}_s(x) = \widehat{X}_s(y)$ for some time s, then $\widehat{X}_t(x) = \widehat{X}_t(y)$ for any $t \geq s$, hence $\widehat{W}^z(x) = \widehat{W}^z(y)$ and $x \overset{\mathscr{A}}{\sim} y$. The opposite direction is bit more difficult. Let $\lambda > 0$ and denote by $(J_i^\lambda, i \geq 1)$ the jumps times of an independent Poisson process with intensity λ. The idea is to study the consecutive partitions C^λ given by

$$i \overset{C^\lambda}{\sim} j \text{ if and only if } \widehat{W}_t^z(J_i^\lambda) = \widehat{W}_t^z(J_j^\lambda).$$

One can show that $C^\lambda = C^\lambda(\infty)$ a.s. where $C^\lambda(\infty)$ is the ancestral partition of sampled individuals $J_i^\lambda, i \geq 1$; see Theorem 2.11. As λ is arbitrary, by letting λ go to ∞ (and thus sampling more and more individuals), we end up with the targeted identity (2.17). The results about the fractal dimensions follow from a general theorem linking the latter to the Laplace exponent of $W^z(\cdot)$, see e.g. Bertoin's lecture notes [17]. □

Under Grey's condition, $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} < \infty$, the process $(\widehat{W}^z(x), x \geq 0)$ is the inverse of a compound Poisson process for any $z \in (0, \infty]$. By choosing $z = \infty$, the latter has for jump measure the quasi-stationary distribution ν_∞. The set of ancestors under Grey's condition is discrete and the partition $\mathscr{A}$ is constituted of i.i.d. families with lengths of law ν_∞. We recover here the result of Theorem 2.13. The following figure provides a schematic representation of the families, their lineages and the process $\widehat{W}^\infty$, under Grey's condition. Recall that multiple coalescences of the lineages can occur.

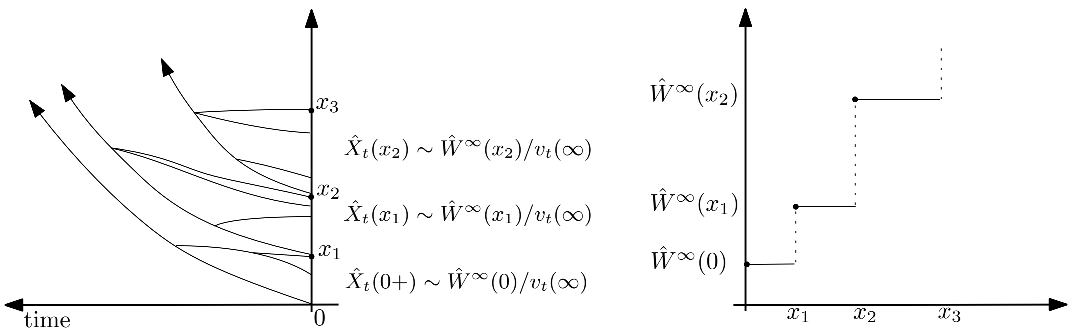


Figure 2.5 – Schematic representation of ancestral families under Grey's condition

Assume from now on that Grey's condition does not hold, i.e. $\int^\infty \frac{\mathrm{d}u}{|\Psi(u)|} = \infty$, see e.g. Example 2.18. In this case the Lévy measure of the subordinator $W^z(\cdot)$ is infinite for all parameter $z > 0$. Its inverse process $(\widehat{W}^z(x), x \geq 0)$ has therefore singular continuous sample paths and any fixed subinterval of $(0,\infty)$ of finite length contains infinitely many microscopic families with positive probability.

In this setting, both cases $\Psi'(\infty)=\infty$ and $\Psi'(\infty)<\infty$ are possible. In the first case, $\mathscr{S}$ is uncountable but has a zero Hausdorff dimension. In the finite variation case, (2.18) can be rewritten as

$$\dim_H(\mathscr{S}\cap[0,x]) = \frac{\gamma}{\gamma+\int_0^\infty h\pi(\mathrm{d}h)} \text{ a.s.}$$

We have only interpreted in terms of the population the jumps sizes of W^z but not their locations (i.e. the values of $\widehat{W}^z$). The latter encodes the rate of escape to ∞ of the ancestral lineages and can be heuristically thought as some ancestral types. In the schematic representation given in Figure 2.5, the divergence of the ancestral lineage of the family (x_2,x_3) is faster than those of (x_1,x_2) and $(0,x_1)$.

2.5.2 A few words on the proofs

In the CSBP setting, i.e. forwards in time, the key tool in order to study the longterm behavior was Grey's martingales, see Chapter 1. The latter only exist thanks to the branching property which we do not have in general with the ALP. We should therefore look for something else in order somehow to replace them. A quick look at its generator, see (2.8) also convinces us that using it directly to get martingales will be difficult. The *search of martingales* will be done by duality arguments and is interesting for itself. We shall encounter several times similar techniques in the next Chapters. Notice the following *bi-duality* diagrams satisfied by the CSBP X and the ALP $\widehat{X}$:

$$X \overset{\text{Laplace duality}}{\longleftrightarrow} u \overset{\text{Siegmund duality}}{\longleftrightarrow} v$$

where $v_t(x) := u_{-t}(x) = \inf\{y>0 : u_t(y) > x\}$ is the renormalisation of the CSBP (at least in the finite variation case, see Theorem 2.B).

$$\widehat{X} \overset{\text{Siegmund duality}}{\longleftrightarrow} X \overset{\text{Laplace duality}}{\longleftrightarrow} u.$$

Heuristically, by exploring these relationships, we will see that the Laplace dual of X (i.e., the cumulant $t\mapsto u_t$ defined in (0.2)) serves as the appropriate renormalization for the ALP.

We start by giving a general theorem for processes in Siegmund duality.

Step 1: Invariant functions of processes in Siegmund duality

We consider here a "general" standard Markov process $X := (X_t, t\geq 0)$ with state space $[0,\infty)$, and denote by $(X_t(y), t\geq 0)$ the process started from $y\in[0,\infty)$. Assume that there exists[9] a Markov process $\widehat{X}$, the Siegmund dual process, such that for any t and x, y

$$\mathbb{P}(X_t(y)\geq x) = \mathbb{P}(\widehat{X}_t(x)\leq y). \tag{2.19}$$

Set

$$\widehat{T}_y^+ := \inf\{t>0 : \widehat{X}_t(x)\geq y\}. \tag{2.20}$$

Our first result shows how to find fundamental martingales for the Siegmund dual process $(\widehat{X}_t(x), t\geq 0)$. The proof of the following theorem is not difficult, it only uses the identity (2.19).

9. Siegmund [135] has shown that this is equivalent to the stochastic monotonicity

Theorem 2.20 (Theorem 4.1 in [F15], Invariant functions of $\widehat{X}$)**.** *Let $(P_t, t \geq 0)$ be the semigroup of the process $(X_t, t \geq 0)$. Let $\theta \in \mathbb{R}$. If μ_θ is a θ-invariant measure, namely a positive Borel measure on $[0, \infty)$ satisfying for any $t \geq 0$, $\mu_\theta P_t = e^{\theta t}\mu_\theta$, then the functions*

$$x \mapsto \mu_\theta([0, x)) \text{ and } x \mapsto \mu_\theta((x, \infty)),$$

provided they are well-defined, are θ-invariant functions, that is to say, functions f_θ such that for any $t \geq 0$ and $x \in [0, \infty)$,

$$\mathbb{E}[f_\theta(\widehat{X}_t(x))] = e^{\theta t} f_\theta(x).$$

Equivalently,

$$(e^{-\theta t} f_\theta(\widehat{X}_t(x)), t \geq 0) \text{ is a martingale.} \tag{2.21}$$

In particular, if the process $(\widehat{X}_t, t \geq 0)$ has no positive jumps and μ_θ is finite on $[0, x)$ for all $x > 0$, then $f_\theta : x \mapsto \mu_\theta([0, x))$ is a well-defined increasing left-continuous function, and for all $y \geq x \geq 0$,

$$\mathbb{E}_x[e^{-\theta \widehat{T}_y}] = \frac{\mu_\theta([0, x))}{\mu_\theta([0, y))}. \tag{2.22}$$

Step 2: Application of Theorem 2.20 to the CSBP and ALP

For any $\theta > 0$, one can check that the map $c_\theta : x \mapsto e^{-\theta \int_1^x \frac{\mathrm{d}u}{\Psi(u)}}$ is completely monotone on $(0, \infty)$. Bernstein-Widder's theorem ensures thus the existence of a Borel measure μ_θ with Laplace transform c_θ,

$$c_\theta(x) = \int_{[0,\infty)} e^{-xy} \mu_\theta(\mathrm{d}y).$$

Note that since $\Psi'(0) > 0$, $\int_0 \frac{\mathrm{d}u}{\Psi(u)} = \infty$, $c_\theta(0) = \infty$ and μ_θ is *infinite*. We see also that $c_\theta(\infty) = \mu_\theta(0) > 0$ if only if $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} < \infty$. We next show that the measure μ_θ is θ-invariant for (P_t) the semigroup of the CSBP X. By applying (0.4), one gets

$$c_\theta(u_t(x)) = e^{-\theta \int_1^{u_t(x)} \frac{\mathrm{d}u}{\Psi(u)}} = e^{\theta t} c_\theta(x).$$

In other words, c_θ is a θ-invariant function for $t \mapsto u_t(x)$ (bidual of $\widehat{X}$). Set for any $x > 0$, $e_x : z \mapsto e^{-xz}$. One has for all $x > 0$,

$$\begin{aligned} \mu_\theta P_t e_x = \int_{[0,\infty)} \mathbb{E}_z(e^{-xX_t}) \mu(\mathrm{d}z) &= \int_{[0,\infty)} e^{-zu_t(x)} \mu_\theta(\mathrm{d}z) \\ &= c_\theta(u_t(x)) = e^{\theta t} c_\theta(x), \end{aligned}$$

where $c_\theta(x) = \int_{[0,\infty)} e^{-xy} \mu_\theta(\mathrm{d}y) = \mu_\theta e_x$. Since this holds for all $x > 0$, we have indeed by injectivity of the Laplace transform $\mu_\theta P_t = e^{\theta t}\mu_\theta$. By our Theorem 2.20, $f_1(x) := \mu_1([0, x))$ is 1-invariant for $\widehat{X}$ and the martingale convergence theorem entails that

$$Z(x) := \lim_{t \to \infty} e^{-t} f_1(\widehat{X}_t(x)) \text{ exists a.s.} \tag{2.23}$$

Step 3: Analysis of f_1

Arguments here are adapted from a work of Pakes [118].

1. One plainly checks that c_1, the Laplace transform of μ_1, is regularly varying at 0 with negative index $-1/\Psi'(0+)$ and by applying a Tauberian theorem, we get the asymptotics of f_1

$$f_1(x) = \mu_1([0,x)) \underset{x\to\infty}{\sim} \frac{c_1(1/x)}{\Gamma(1+1/\Psi'(0+))}. \tag{2.24}$$

2. Using next the regular variation property, we get that $c_1^{-1}(e^t)$ is equivalent to $u_t(z)$ as t goes to ∞ up to a multiplicative constant. By (2.23) and (2.24), we obtain, almost surely, as t goes to ∞, by the regular variation of c_1^{-1}

$$\begin{aligned}\widehat{X}_t(x) \sim f_1^{-1}(e^t Z(x)) \sim C/c_1^{-1}(e^t Z(x)) &\sim C' \frac{c_1^{-1}(e^t)}{c_1^{-1}(e^t Z(x))} \frac{1}{u_t(z))} \\ &\sim \widehat{W}^z(x)/u_t(z)\end{aligned}$$

for some constants $C, C' > 0$ and where we have set $\widehat{W}^z(x) := C' Z(x)^{-\Psi'(0+)}$.

Step 4: Identification of the limit and convergence of the flow

We identify the law of the limit for a fixed x, by showing directly that the subordinator $X_t(\cdot/u_t(z))$ converges in finite dimensional law towards a subordinator $W^z(\cdot)$ with Laplace exponent κ_z given in (2.15). This is seen for instance by writing

$$\mathbb{E}[e^{-\theta X_t(y/u_t(z))}] = e^{-y u_t(\theta)/u_t(z)} \underset{t\to\infty}{\longrightarrow} e^{-y\kappa_z(\theta)}$$

where we used (2.16). The Siegmund duality relationship then enables us to conclude that the process $\widehat{W}^z(\cdot)$ has the targeted finite-dimensional marginal laws. The task of "putting x in the almost-sure" is merely technical. □

2.6 Comments

Literature and a remaining question

The long-term behavior of the ALP process in the critical and supercritical cases has been left unaddressed. In the critical setting, it can be established that $u_t(z)\widehat{X}_t(x)$ converges *in law* towards an exponential random variable (with a parameter only depending on z). Another possible angle of study of the critical case would be to seek a functional limit theorem as what has been done for CBIs in [F19], see Chapter 1.3. I have no clue however whether some interesting phenomena would arise or not.

I wish to mention that Siegmund dual processes of branching processes have appeared under different names in the literature. In particular, I should highlight the work of Pakes [118], see also the references therein, where they are called dual Markov branching processes. Several interesting limit theorems have been established in this work. To the best of my knowledge, those processes were however not connected to the genealogy of the branching population. In the setting of the Fleming-Viot processes, such a genealogical interpretation of Siegmund duality was however already given through the notion of fixation line. We refer to the seminal paper of Olivier Hénard, see [75]. Siegmund duality also appears very often in the theory of interacting particle systems, see e.g. Clifford and Sudbury [40] for one of the first work in this direction.

Ouverture

A potential avenue of work, which I hope to consider at some point, is the study of continuous-state branching-coalescing (braco) processes. The basic idea is to mix the dynamics of the CSBPs with that of the ALPs; see Figure 2.3, so that we compose classical branching events, of “jump” type, with coalescence event of “constant stretch” type. The process obtained can be also thought as a branching process with a certain form of (density-dependent) catastrophes. The discrete counterpart of this model has been recently introduced by Carrance et al. [35]. Among other things, conditions are given for the random trees subject to those coalescences to converge towards the continuous random tree.

Several interesting phenomena could occur in those “braco” processes, also by looking directly in the continuous limit. We might for instance wonder if a non trivial *compensated* path-valued process can be defined along this way.

Part II

Branching processes with interaction and Laplace duality

We have seen in Part I of this document, how the branching property allows both the study of the Markov process recording the size of the population (for classifying e.g. the extinction/survival behaviors) and the genealogy of the associated population model. In this second part, we are going to study and define generalizations of CSBPs in which a dynamics of interaction (such as competition) between individuals is taken into account. This will break down the branching property since heuristically two subpopulations will interact and not be independent. The loss of the branching property renders of course the study much more involved. The objective in this part is to study the processes themselves and not the genealogy. We refer the reader to the works cited in the general introduction for a study at the level of the genealogy.

In Chapter 3, based on [F8] and [F20], we will study the *logistic* CSBPs. Additionally to the reproduction, the latter take into account a competition term which is modelled along a negative quadratic drift. They have been introduced by Amaury Lambert in [92]. From a heuristic point of view, the negative drift can be interpreted as duels in which only one individual wins. In this chapter, we will be mainly interested in the behavior of the process near ∞. We shall see that despite the competition, some branching mechanism still enable the process to explode in finite time. Our main interest will be to discuss what happens if one starts an explosive process from ∞. We will find an explicit condition for the process to make excursions away from infinity and study its local time at ∞.

In Chapter 4, extirped from [F17], we will define a broader generalization of CSBPs in which an additional noise term is considered. This will represent a certain notion of random collisions. The idea here is that, besides the classical branching dynamics, two individuals are picked at random in the current population, continuously in time, collide and disappear, leaving behind a *small* random mass of individuals. The aim here is not to classify the behaviors at the boundary ∞ but to study the first passage times below a level (including the time of extinction) as well as the longterm behavior of the process (survival/extinction/stationarity).

In both chapters, our study will rely on duality relationships. Chapter 3 starts with the observation that a Laplace duality holds between a logistic CSBP Z and a Feller diffusion U with a generalized drift (given by the branching mechanism). The Laplace duality (either at the level of the semigroup or at the generator) will somehow replace the branching property. Siegmund duality for one-dimensional diffusions will also be cornerstone in both chapters. We dedicate to it an independent section, see Chapter 3.4. We shall indeed exploit in several ways the following duality diagram

$$Z \overset{\text{Laplace dual}}{\longleftrightarrow} U \overset{\text{Siegmund dual}}{\longleftrightarrow} V.$$

where U and V are some diffusions.

In Chapter 4, we will see that the processes with collisions are in Laplace duality with a broader class of diffusions than Feller diffusions with generalized drift. More precisely, the diffusion coefficient of U will be given by the Lévy-Khintchine function governing the collisions.

The world of regular [10] one-dimensional diffusions being totally understood since the sixties and the works of Feller, see [60], the duality relationships will be an entrance door for studying in a complete and fairly explicit manner the processes. For the LCSBPs, one-to-one correspondences will be made between the nature of the boundaries of the processes. Feller's tests will transfer to the framework of the càdlàg process Z. For the processes with collision, we shall connect among other things the first passage-times of the diffusion V to those of Z.

10. in the sense with smooth coefficients

CHAPTER 3

Logistic CSBP, Laplace duality and reflection at infinity

Summary.

In this chapter, we study the logistic CSBPs in which besides classical reproduction events, pairwise fights between individuals occur. The latter are represented by incorporating a quadratic negative drift to the dynamics of the population size. These processes were first introduced by Lambert in [92]. Lambert's approach in [92] relies on a detailed qualitative analysis of certain Riccati ordinary differential equations, which are quadratic in the unknown function. Here, we will adopt a different point of view.

We start by defining *minimal* LCSBPs (i.e. processes stopped after hitting ∞) from a martingale problem and to study their explosion. Later on we establish a Laplace duality relationship between the LCSBP Z and a diffusion U belonging to a certain class of generalized Feller diffusions. The process U will somehow play the role of the cumulant u of the classical CSBP. Feller's boundary integral tests, recalled in Annex C.2.4, will allow us to classify the boundaries of the LCSBP. The duality will also help us to spotlight what can happen at the boundary ∞ of a logistic CSBP whose branching Lévy measure has very heavy tails (typically slowly varying at ∞). We will indeed find a necessary and sufficient condition for a logistic CSBP to have its boundary infinity respectively entrance, regular or exit. In the regular case, we will construct an extension of the minimal process which can leave continuously the boundary ∞ and returns. The point ∞ will be moreover regular for itself and regular reflecting. The tasks to study the law of the explosion time and to describe the local time and the excursion measure away from ∞ will also be addressed. In order to do so, we shall use a *second* duality relationship and introduce the process V obtained as the Siegmund dual of the Laplace dual U.

In 1838, Verhulst has introduced the *logistic* function to model population in situations where resources are limited, see [137]. His aim was to develop a model preventing the unrealistic exponential growth predicted by Malthus' theory. The function is the solution of Ricatti's ordinary differential equation

$$\mathrm{d}z_t = \gamma z_t \mathrm{d}t - \frac{c}{2} z_t^2 \mathrm{d}t. \tag{3.1}$$

The logistic CSBP, in short LCSBP, introduced in [92], can be thought as the random analogue of $(z_t)_{t\geq 0}$, where, informally speaking, the Malthusian growth $\gamma z_t dt$ is replaced by the full dynamics of a continuous-state branching process.

Our objective is to study the possible longterm behaviors of LCSBPs for most general branching mechanisms.

The techniques designed by Lambert in [92] were based on a comprehensive qualitative analysis of certain nonlinear ordinary differential equations. We will take an other point of view and see new phenomena arising at the boundary ∞.

3.1 Minimal logistic CSBPs and their explosion

Let Ψ be a branching mechanism, recall that it has the following Lévy-Khintchine form

$$\Psi(z) = -\lambda + \frac{\sigma^2}{2} z^2 + \gamma z + \int_0^{+\infty} \left(e^{-zx} - 1 + zx \mathbb{1}_{\{x \leq 1\}} \right) \pi(\mathrm{d}x) \tag{3.2}$$

with $\lambda \geq 0, \gamma \in \mathbb{R}$, $\sigma \geq 0$, and π a Lévy measure on $(0, \infty)$. The parameter λ here can be seen as a mass at ∞ for the Lévy measure π (the CSBP with a mechanism such that $\Psi(0) = -\lambda < 0$ jumps from z to ∞ at rate λz).

Recall $\mathscr{L}^{\Psi}$ the generator of the CSBP, see Annex A.2.2, and define

$$\mathscr{L} f(z) := \mathscr{L}^{\Psi} f(z) - \frac{c}{2} z^2 f'(z), \ \forall z \geq 0. \tag{3.3}$$

Notice the negative quadratic drift term. The operator $\mathscr{L}$ will be the infinitesimal generator of our LCSBPs.

As usual, there are several ways to introduce such a process. We choose here to define it through a martingale problem. The approach of stochastic equations will be chosen in the next chapter.

Definition 3.1. A *minimal* LCSBP is a càdlàg Markov process $(Z_t^{\min}, t \geq 0)$ on $[0, \infty]$ with 0 and ∞ *absorbing*, satisfying the following martingale problem **(MP)**. For any function $f \in C_c^2((0, \infty))$, the process

$$t \in [0, \infty) \mapsto f(Z_t^{\min}) - \int_0^t \mathscr{L} f(Z_s^{\min}) \, \mathrm{d}s \tag{3.4}$$

is a martingale.

Let $\zeta := \inf\{t \geq 0; Z_t^{\min} \notin (0, \infty)\}$. By definition the minimal process remains at the boundary once it has reached it. In particular, by setting $\zeta_\infty := \inf\{t \geq 0, Z_t^{\min} = \infty\}$, on the event $\{\zeta = \zeta_\infty\}$, we have that $Z_t^{\min} = \infty$ for any $t \geq \zeta_\infty$.

Theorem 3.2 (Lemmas 4.1 and 4.2 in [F8])**.** *There exists a unique minimal LCSBP* $(Z_t^{\min}, t \geq 0)$.

Sketch of proof. Following an idea already present in [92], the minimal process $Z^{\min}$, starting from $z > 0$, can be constructed as follows. Let Y be a spectrally positive Lévy process with Laplace exponent Ψ, starting from $z > 0$, and set

$$R_t = Y_t - \frac{c}{2}\int_0^t R_s \mathrm{d}s, \quad t \mapsto \theta_t := \int_0^{t\wedge\sigma_0} \frac{\mathrm{d}s}{R_s} \in [0,\infty],$$

with $\sigma_0 := \inf\{t > 0 : R_t < 0\}$. We refer for instance to Sato's book [129, Chapter 3.17] for a study of the process R, called generalized Ornstein-Uhlenbeck (GOU). Notice that its generator is of the form $\mathscr{L}^R f(z) := \mathrm{L}^\Psi f(z) - \frac{c}{2} z f'(z)$, where L^Ψ is the generator of Y. The identity $z\mathscr{L}^R f(z) = \mathscr{L} f(z)$, together with Volkonski's theorem on time-changes of Markov processes, see also the references in Annex A.2.3, ensures that $(Z_t^{\min}, t \geq 0)$ defined by

$$Z_t^{\min} = \begin{cases} R_{C_t} & 0 \leq t < \theta_\infty, \\ 0 & t \geq \theta_\infty \text{ and } \sigma_0 < \infty, \\ \infty & t \geq \theta_\infty \text{ and } \sigma_0 = \infty. \end{cases}$$

with $C_t := \inf\{u \geq 0 : \theta_u > t\}$, is a minimal LCSBP. In the case without competition, we recover Lamperti's time-change of CSBPs, see Annex A.2.3. By applying standard techniques (as for instance reasoning by time-change again), it is clear that there is a *unique* solution to the stopped martingale problem (MP). □

The first main result is the following necessary and sufficient condition for explosion, i.e. for accessibility of ∞.

Theorem 3.3 (Accessibility of ∞, Theorem 3.1 in [F8])**.** *Assume $c > 0$. The boundary ∞ is accessible for $(Z_t^{\min}, t \geq 0)$ if and only if*

$$\mathcal{E} := \int_0^{x_0} \frac{1}{x} \exp\left(\frac{2}{c}\int_x^{x_0} \frac{\Psi(u)}{u}\mathrm{d}u\right)\mathrm{d}x < \infty, \text{ for some (and then for all) } x_0 > 0. \quad (3.5)$$

Of course, when $\lambda > 0$, one has explosion and $\mathcal{E} < \infty$, but this is not a necessary condition. There exist mechanisms Ψ for which explosion occurs “continuously" and not by a single jump to ∞, see the forthcoming Example 3.5.

Theorem 3.3 can be for instance established by studying the finiteness of the perpetual integral $\theta_\infty = \int_0^\infty \frac{\mathrm{d}s}{R_s}$, on the event $\{\sigma_0 = \infty\}$. It turns out that conditionally on $\{\sigma_0 = \infty\}$, the integral converges as soon as the process R is transient (goes to ∞). The condition $\mathcal{E} < \infty$ is known to be the necessary and sufficient condition for transience of the GOU process, this is a result due to Shiga [134]. This is also telling us that no minimal LCSBPs are transient without being exploding. In other words, the population cannot grow indefinitely when there is quadratic competition.

Now that we know when the boundary ∞ is accessible or not, we would like to see whether we can start the process from it. They are in general four possibilities:

— The process cannot hit ∞ but can be started from it (∞ is an entrance).

— The process hits ∞ (explodes) and gets stuck at it (∞ is an exit).

— The process explodes and gets back in the half-line at a future time (∞ is regular).

— The process neither can be started from ∞ nor visits it (∞ is natural).

This classification is reexplained in Annex C.1 in the context of one-dimensional diffusions. The three first cases will occur in our setting. The last case will not.

The third case requires defining a process past its explosion. In the sequel, we say that a Markov process $(Z_t, t \geq 0)$ extends the minimal process if $(Z_t, t \geq 0)$ takes its values in $[0,\infty]$ and

$$(Z_{t\wedge\zeta_\infty}, t \geq 0) \overset{\mathcal{L}}{=} (Z_t^{\min}, t \geq 0).$$

Note that such extended processes certainly exist if we authorize elementary return processes which are restarting after explosion from states in $(0,\infty)$. We will however only consider *continuous* and *instantaneous* Feller extension $(Z_t, t \geq 0)$, i.e. for which $Z_t \to \infty$, almost surely, as $t \to \zeta_\infty+$ (if ∞ is accessible) and $\mathbb{P}_\infty(T = 0) = 1$ with $T := \inf\{t > 0, Z_t < \infty\}$. The boundary is moreover said to be *reflecting*, if $\mathbb{P}_\infty(Z_t = \infty) = 0$ for any $t > 0$, (equivalently the set $\{t > 0, Z_t = \infty\}$ has a Lebesgue measure zero).

A classical approach to define a Markov extension of a minimal process is the so-called *Itô's synthesis*. First an excursion measure is constructed and then the extended process is built by concatenation of the excursions along a Poisson point process. We refer the reader for instance to Blumenthal's book [30, Chapter V]. The basic ingredients needed for this program to have a chance to work are the resolvent of the minimal process and the law of the explosion time. However, we do not yet have much information available on the latter. We will choose another approach. The study of the excursions away from ∞ will be done in a second time. The extended Feller processes will be constructed by limits with the help of a Laplace duality relationship.

3.2 LCSBPs, Laplace duality and boundary classification

For any $x, z \in (0,\infty)$, set

$$e_x(z) = e_z(x) = e^{-xz}.$$

Those functions will play a role in all Part II. The starting point of the study is the following identity:

$$\mathscr{L} e_x(z) = \mathscr{A} e_z(x), \tag{3.6}$$

with $\mathscr{A}$ the operator defined on $C^2(0,\infty)$ as follows: for any $g \in C^2(0,\infty)$ and $x \in (0,\infty)$,

$$\mathscr{A} g(x) := \frac{c}{2} x g''(x) - \Psi(x) g'(x). \tag{3.7}$$

The relation (3.6) is very easy to check, we do not explain it here as we will do explicitly a similar calculation in the next chapter. It is noteworthy to observe that the *pseudo-differential* operator $\mathscr{L}$ is mapped into a *local* operator. Having a duality at the level of generators is just one step in the quest of establishing a duality for the semigroup, see the discussion in Annex D.2. Indeed the operator $\mathscr{A}$, without specifying the domain, only characterizes the diffusion U stopped when exiting $(0,\infty)$, see Annex C.2.1, that is the weak solution to the stochastic differential equation

$$\mathrm{d}U_t = \sqrt{cU_t}\,\mathrm{d}B_t - \Psi(U_t)\mathrm{d}t,\ U_0 = x, \tag{3.8}$$

with $(B_t, t \geq 0)$ a Brownian motion. Notice that when $c = 0$, the equation (3.8) shrinks into the ordinary differential equation solved by the cumulant $(u_t, \geq 0)$, see (A.15).

When $c > 0$, Feller's tests, see Annex C.2.4, can be applied to see the possible behaviors of U at the boundaries. In particular, recall $\mathcal{E}$ in (3.5), we have $\mathcal{E} = \frac{c}{2} M_U(0, x_0]$, where M_U is the speed measure of U. Feller's tests simplify nicely and, recalling $\lambda = -\Psi(0)$, the boundary 0 is accessible if and only if $2\lambda/c < 1$, it is an exit if $\mathcal{E} = \infty$, regular if $\mathcal{E} < \infty$ and an entrance if $2\lambda/c \geq 1$.

The regular case for 0 is exactly mirroring the fact that perhaps other processes than the minimal LCSBP are solving the (unstopped) martingale problem (**MP**). In other words, extensions of the minimal process may exist.

We sum up in the next theorem, the results obtained in [F8] on the extensions of the minimal process, as well as their behaviors near the boundary 0 (extinction) when the boundary ∞ is non-absorbing.

Theorem 3.4 (Theorems 3.1, 3.3, 3.4 and 3.9 in [F8]).

i) *Feller extensions: There exists a Feller*[1] *process* $(Z_t, t \geq 0)$ *on* $[0, \infty]$ *with no negative jumps, extending the minimal process* $Z^{\min}$*, such that for any* $x, z \in (0, \infty]$ *and* $t \geq 0$,

$$\mathbb{E}_z[e^{-xZ_t}] = \mathbb{E}_x[e^{-zU_t}], \tag{3.9}$$

where $(U_t, t \geq 0)$ *is the weak solution to* (3.8) *with boundary conditions at* 0 *given in correspondence with that of* Z *at* ∞ *as in Table 3.1.*

Integral condition	Boundary of U	Boundary of Z
$\mathcal{E} = \infty$	0 exit	∞ entrance
$\mathcal{E} < \infty$ and $2\lambda/c < 1$	0 regular absorbing	∞ regular reflecting
$2\lambda/c \geq 1$	0 entrance	∞ exit

Table 3.1 – Boundaries ∞ and 0 of Z, U.

ii) *Extinction:*

(a) If $2\lambda/c < 1$ *(i.e.* Z *has the boundary* ∞ *either entrance or regular reflecting), then*

— Z *converges towards* 0 *a.s. if and only if* $\Psi(\infty) = \infty$ *(i.e. the pure CSBP(*Ψ*) is not immortal)*

— Z *gets absorbed at* 0 *a.s. if and only if* $\Psi(\infty) = \infty$ *and* $\int^{\infty} \frac{\mathrm{d}x}{\Psi(x)} < \infty$.

b) If $2\lambda/c \geq 1$ *(i.e.* Z *has the boundary* ∞ *exit), then*

— *If* $\Psi(\infty) = \infty$ *then*

$$\mathbb{P}_z(Z_t \underset{t\to\infty}{\longrightarrow} 0) = \frac{\int_0^{\infty} e^{-zu}\frac{1}{u}\exp\left(-\int_{\theta}^{u}\frac{2\Psi(v)}{cv}\mathrm{d}v\right)\mathrm{d}u}{\int_0^{\infty}\frac{1}{u}\exp\left(-\int_{\theta}^{u}\frac{2\Psi(v)}{cv}\mathrm{d}v\right)\mathrm{d}u} \in (0,1)$$

— Z *gets absorbed at* 0 *with positive probability if and only if* $\Psi(\infty)$ *and* $\int^{\infty} \frac{\mathrm{d}u}{\Psi(u)} = \infty$.

Integral condition	Boundary of U	Boundary of Z
$\int^{\infty} \frac{\mathrm{d}x}{\lvert\Psi(x)\rvert} = \infty$	∞ natural	0 natural
$\int^{\infty} \frac{\mathrm{d}x}{\lvert\Psi(x)\rvert} < \infty$	∞ entrance	0 exit

Table 3.2 – Boundaries ∞ and 0 of U, Z.

1. Here Feller means that the semigroup maps continuous bounded functions on $[0, \infty]$ into themselves

We summarize the four possible behaviors of the process Z at ∞ in Figure 3.1 along with ultra-simplistic ugly drawings[2]. We distinguish the two cases $\lambda = 0$ and $\lambda > 0$. In the first setting, when there is explosion it is due to an accumulation of large jumps (the process climbs to ∞) in the second case the process is sent at ∞. The possible behaviors at 0 when ∞ is not an exit and the branching part is not immortal (i.e. the pure CSBP can go towards 0) are summarized by Figure 3.2.

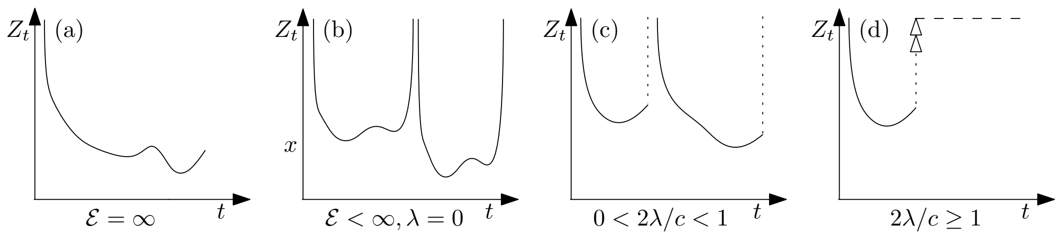


Figure 3.1 – Schematic representation of the four behaviors at ∞.

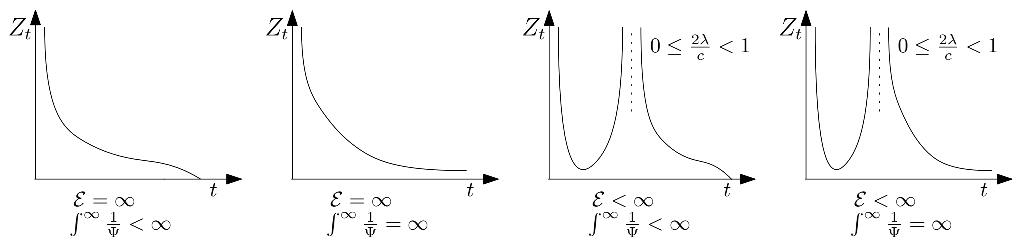


Figure 3.2 – Schematic representation of the four behaviors at 0 when $0 \leq 2\lambda/c < 1$.

Example 3.5.

1. Consider $\alpha \in (0, 2]$, $\alpha \neq 1$ and $\Psi(z) = (\alpha - 1)z^{\alpha}$. Since $\int_0 \frac{|\Psi(z)|}{z} \mathrm{d}z < \infty$, we have that $\mathcal{E} = \infty$ and ∞ is an entrance boundary (case (a) in Figure 3.1). For any $t \geq 0$, $z \in [0, \infty]$ and $x \in [0, \infty)$
$$\mathbb{E}_z(e^{-xZ_t}) = \mathbb{E}_x(e^{-zU_t}) \text{ with } \mathrm{d}U_t = \sqrt{cU_t}\mathrm{d}B_t + (1 - \alpha)U_t^{\alpha}\mathrm{d}t,\ U_0 = x,$$
the boundary 0 of $(U_t, t \geq 0)$ being an exit. Recall that when $\alpha \in (0, 1)$, the CSBP without competition explodes, see Annex A, so that here competition prevents explosion.
2. Let $\lambda > 0$ and $\Psi(x) = -\lambda$ for all $x \geq 0$.
   i) If $\frac{2\lambda}{c} < 1$ then ∞ is regular reflecting (case (c) in Figure 3.1). For any $t \geq 0$, $z \in [0, \infty]$ and $x \in [0, \infty)$
   $$\mathbb{E}_z(e^{-xZ_t}) = \mathbb{E}_x(e^{-zU_t^0}) \text{ with } \mathrm{d}U_t^0 = \sqrt{cU_t^0}\mathrm{d}B_t + \lambda\mathrm{d}t,\ U_0^0 = x,$$
   the boundary 0 of $(U_t^0, t \geq 0)$ being regular absorbing.
   ii) If $\frac{2\lambda}{c} \geq 1$ then ∞ is an exit (case (d) in Figure 3.1). For any $t \geq 0$, $z \in [0, \infty]$ and $x \in (0, \infty)$
   $$\mathbb{E}_z(e^{-xZ_t}) = \mathbb{E}_x(e^{-zU_t}) \text{ with } \mathrm{d}U_t = \sqrt{cU_t}\mathrm{d}B_t + \lambda\mathrm{d}t,\ U_0 = x,$$
   the boundary 0 of $(U_t, t \geq 0)$ being an entrance.

2. of course sample paths do not ressemble to that

This phase transition between exit and regular reflecting is reminiscent to that for the so-called fast-fragmentation-coalescence processes obtained by Kyprianou et al. [89]. We shall return to this in Chapter 5.

We now provide an example with ∞ regular and no single jump to ∞.

3. Consider $\lambda = 0$, $\sigma \geq 0$, $\gamma \in \mathbb{R}$ and set $\pi(\mathrm{d}u) = \frac{\alpha}{u(\log u)^{\beta+1}} \mathbb{1}_{\{u \geq e\}} \mathrm{d}u$ for some $\alpha > 0$, $\beta > 0$. The branching mechanism satisfies $\Psi(x) \underset{x \to 0+}{\sim} -\alpha/\log(1/x)^{\beta}$.
   - i) If $\beta > 1$ or $\beta = 1$ and $\frac{2\alpha}{c} \leq 1$ then $\mathcal{E} = \infty$ and ∞ is an entrance boundary (case (a) in Figure 3.1) for Z, 0 is an exit for U.
   - ii) If $\beta = 1$ and $\frac{2\alpha}{c} > 1$ then $\mathcal{E} < \infty$ and ∞ is a regular reflecting boundary (case (b) in Figure 3.1) for Z, 0 is regular absorbing for U.
   - iii) If $\beta \in (0,1)$, then $\mathcal{E} < \infty$ and ∞ is a regular reflecting boundary (case (b) in Figure 3.1), 0 is regular absorbing for U.

   Note the phase transition occurring when $\beta = 1$ between the entrance and regular types.

We plot here a simulation of a sample path of U whose 0 is a regular reflecting. Note that the dual process of the LCSBP with ∞ regular *reflecting* is the diffusion U with 0 regular *absorbing*, see Theorem 3.4.

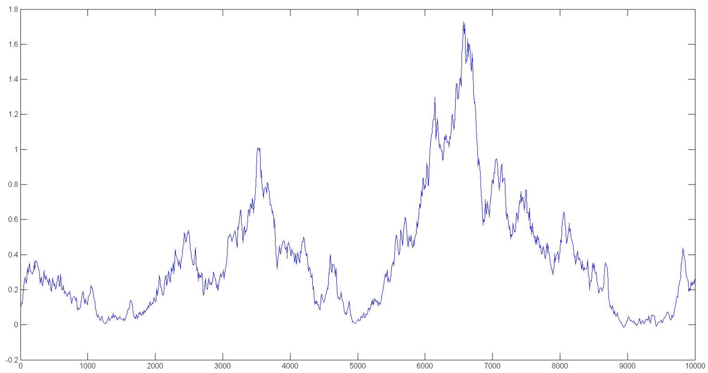


Figure 3.3 – The diffusion U of Example 3.5-3.(iii) with 0 regular reflecting (β close to 0).

When $-\Psi$ is the Laplace exponent of a subordinator (hence the associated CSBP has non-decreasing sample paths) and the boundary ∞ is regular or entrance, the logistic CSBP may have a stationary distribution. The next theorem yields a necessary and sufficient condition for a stationary distribution to exist and provides its Laplace transform.

Theorem 3.6 (Stationarity, Theorem 3.7 in [F8]). *Assume $\Psi(\infty) = -\infty$, in this case Ψ takes the form*

$$\Psi(z) = -\lambda - \delta z - \int_0^{\infty} (1 - e^{-zu}) \pi(\mathrm{d}u)$$

*with $\lambda \geq 0$, $\delta \geq 0$ and $\int_0^{\infty} (1 \wedge u) \pi(\mathrm{d}u) < \infty$. Assume $0 \leq \frac{2\lambda}{c} < 1$ and define the condition **(A)** as follows*

$$\textbf{(A)} : (\delta = 0 \text{ and } \overline{\pi}(0) + \lambda \leq c/2).$$

- *If **(A)** is satisfied then $(Z_t, t \geq 0)$ converges in probability to 0.*

- *If* ***(A)*** *is not satisfied then* $(Z_t, t \geq 0)$ *converges in law towards the distribution supported on* $(\frac{2\delta}{c}, \infty)$ *whose Laplace transform is*

$$L : x \in \mathbb{R}_+ \mapsto \mathbb{E}[e^{-xZ_\infty}] := \frac{\int_x^\infty \exp\left(\int_{x_0}^y \frac{2\Psi(z)}{cz}\mathrm{d}z\right)\mathrm{d}y}{\int_0^\infty \exp\left(\int_{x_0}^y \frac{2\Psi(z)}{cz}\mathrm{d}z\right)\mathrm{d}y}. \tag{3.10}$$

Remark 3.7. Note that the formula (3.10) does not depend on $x_0 > 0$. Moreover, the condition for the existence of a non-degenerate stationary distribution can be rephrased as follows. The condition **(A)** is not satisfied if and only if at least one of the following holds

$$\lim_{u\to\infty} \tfrac{\Psi(u)}{u} = -\delta \neq 0,\ \pi((0,1)) = \infty,\ \overline{\pi}(0) + \lambda > \tfrac{c}{2}.$$

The duality is again a central tool in the proof of Theorem 3.6. Explanations on how it is used will be given in Chapter 4, in which we will study the stationary distribution of another class of processes.

Construction of an extension by a limiting procedure

Elements of proof. We sketch here the construction of the extensions of the minimal process using the duality. This is done by taking limits in a sequence of non-explosive LCSBPs.

Step 1: We truncate all jumps of the LCSBPs of size larger than k by putting them at level k. In other words, we look at the sequence of LCSBPs $(Z^{(k)}, k \geq 1)$ with fixed competition parameter c but branching mechanism Ψ_k whose Lévy measures are

$$\pi_k(\mathrm{d}u) := \pi(\mathrm{d}u)\mathbb{1}_{u<k} + \overline{\pi}(k)\delta_k.$$

Clearly $|\Psi_k'(0+)| < \infty$ and the LCSBPs(Ψ_k, c) are not explosive. In this case, Ethier-Kurtz's sufficient conditions, see Annex D.2 can be checked and one can go "safely" from the duality relationship (3.6), to the following one

$$\mathbb{E}_z[e^{-xZ_t^{(k)}}] = \mathbb{E}_x[e^{-zU_t^{(k)}}], \tag{3.11}$$

with $U^{(k)}$ strong solution to

$$\mathrm{d}U_t^{(k)} = \sqrt{cU_t^{(k)}}\mathrm{d}B_t - \Psi_k(U_t^{(k)}). \tag{3.12}$$

Notice now that 0 is an exit for all $U^{(k)}$ (since $\Psi_k(0) = 0$ and $|\Psi_k'(0+)| < \infty$). Letting z go to ∞ in (3.11) yields that the processes $Z^{(k)}$ have all ∞ as entrance

$$\mathbb{E}_\infty[e^{-xZ_t^{(k)}}] = \mathbb{P}_x(U_t^{(k)} = 0) > 0.$$

Step 2. Notice that $\Psi_k \geq \Psi_{k+1}$, so that by a comparison theorem, see e.g. [127, Theorem IX.3.7], for a fixed Brownian motion B, the processes, started from a same value, satisfy a.s.

$$U_t^{(k+1)} \leq U_t^{(k)} \text{ for all } t \geq 0. \tag{3.13}$$

This ensures that the sequence of processes $(U^{(k)}, k \geq 1)$ is converging towards a process U. By the duality (3.11), we see then that $(Z^{(k)})$ converges in the finite dimensional sense. We can also easily check that the convergence of the semigroups is uniform, this entails that the convergence holds in the Skorohod space, see Ethier and Kurtz [56, Theorem 2.5 page 167]. Hence we have, as k goes to ∞,

$$(Z_t^{(k)}, t \geq 0) \Longrightarrow (Z_t, t \geq 0) \tag{3.14}$$

for some process Z valued with state-space $[0, \infty]$, endowed for instance with the compact metric $d(x,y) = |e^{-x} - e^{-y}|$ with $e^{-\infty} = 0$.

Step 3. We show that the limiting processes U and Z are extensions of the minimal processes. The argument relies on the uniform convergence of the generators of $Z^{(k)}$ and $U^{(k)}$ towards $\mathscr{L}$ and $\mathscr{A}$, respectively, on C^2 functions with compact support. This is straightforward to verify. Consequently, the processes U and Z, when stopped upon exiting $(0, \infty)$, satisfy the martingale problems of the minimal diffusion and the minimal LCSBP. Given that these martingale problems are well posed, see Annex C.2.1 for the diffusion, and Theorem 3.2 for the LCSBP, we deduce that Z and U indeed extend $Z^{\min}$ and the minimal diffusion with generator $\mathscr{A}$.

Step 4. The boundary ∞ of Z will be related to the boundary 0 of the limiting process U. We study the behavior at 0 of U. Recall that 0 is accessible if and only if $0 \leq 2\lambda/c < 1$. It is an exit boundary if $\mathcal{E} = \infty$ and a regular boundary if $\mathcal{E} < \infty$. By (3.13), $U_t \leq U_t^{(k)}$ for any time $t \geq 0$. Since $U^{(k)}$ cannot escape from 0, this implies that U is absorbed at 0. Hence, 0 is regular *absorbing* (meaning that although the process can theoretically continue after hitting 0 it is forced to stay there). In the case $2\lambda/c \geq 1$, 0 is an entrance. In any case, plugging the convergence (3.14) in the "pre-duality" result (3.11) provides a duality at the level of semigroups for the process Z:

$$\mathbb{E}_z[e^{-xZ_t}] = \mathbb{E}_x[e^{-zU_t}]. \tag{3.15}$$

It remains to explore the behavior of Z at ∞ and explain Table 3.1. By letting z go to ∞ in (3.15), we get

$$\mathbb{E}_\infty[e^{-xZ_t}] = \mathbb{P}_x(U_t = 0) = \mathbb{P}_x(\tau_0 \leq t), \tag{3.16}$$

with $\tau_0 := \inf\{t > 0 : U_t = 0\}$.

Starting from $z \in (0, \infty)$, the process Z coincides in law up to explosion with $Z^{\min}$. Since in the case $\mathcal{E} = \infty$ the minimal process does not explode, the process Z stays also finite. Moreover 0 is an exit boundary for U, hence the right-hand side of (3.16) is strictly positive and the boundary ∞ is an *entrance*.

In the second case, $\frac{2\lambda}{c} < 1$ and $\mathcal{E} < \infty$, $Z^{\min}$ explodes with positive probability and so does Z since it is an extension. Since U has 0 regular *absorbing*, then again (3.16) is not degenerated at 0. Moreover Z has ∞ regular *reflecting* since

$$\mathbb{P}_z(Z_t < \infty) = \lim_{x \to 0} \mathbb{E}_z[e^{-xZ_t}] = \lim_{x \to 0} \mathbb{E}_x(e^{-zU_t}) = 1.$$

In the last case $\frac{2\lambda}{c} \geq 1$, 0 being an entrance, τ_0 is infinite a.s. and the right-hand is identically 0 so that Z has its boundary ∞ *exit*. □

To sum up, the Laplace duality (3.15) can be used as a representation of the semigroup of the LCSBP Z in terms of that of the diffusion U in a manner analogous to the classical CSBP and its cumulant u. Notice that the probability entrance law of the process from ∞, see (3.16), in other words "what happens in the process past explosion" is entirely encoded in the law of τ_0, the first hitting time of 0 of U.

The extended process Z reflected at its boundary ∞ has been constructed as a limit of LCSBPs whose boundaries ∞ are all of entrance type. In particular, this construction did not give any information on the excursions away from infinity. We will see in the next section that when the boundary ∞ of Z is regular reflecting, ∞ is also regular for itself, namely the process starting from ∞ returns at ∞ instantaneously. This entails that the process Z has a non-degenerate local time at ∞. Its study is the aim of the next section.

3.3 Local time, excursion measure and biduality

The aim here is to push further the analysis of the process Z by studying the laws of its first explosion time and last but not least that of its local time at ∞ when ∞ is a regular reflecting boundary. The statements below can be found in [F20].

We will employ a second duality relationship and introduce the Siegmund dual of process U: namely, the process V satisfying, for any $x, y \in (0,\infty)$ and $t \geq 0$,

$$\mathbb{P}_x(U_t < y) = \mathbb{P}_y(x < V_t). \tag{3.17}$$

We shall appeal locally here to some results about Siegmund duality for one-dimensional diffusions processes stated in the forthcoming Section 3.4.

We summarize both dualities in the following diagram:

$$Z \overset{\textbf{Laplace dual}}{\longleftrightarrow} U \overset{\textbf{Siegmund dual}}{\longleftrightarrow} V. \tag{3.18}$$

We call V the *bidual* process of Z. In essence, a duality relation allows us to understand the process through the time-reversal of its dual, as outlined in D.2. By applying two dualities consecutively, we link therefore the two processes Z and V in the same time direction. Indeed, denoting by $\mathbb{e}_x$ an exponential random variable with parameter x independent of Z. We get by combining the dualities at the level of semigroups (3.15) and (3.17):

$$\mathbb{E}_z[e^{-xZ_t}] = \mathbb{P}_z(\mathbb{e}_x > Z_t) = \int_0^\infty z e^{-zy} \mathbb{P}_y(V_t > x) \mathrm{d}y. \tag{3.19}$$

The preceding duality relationships allow us to connect many properties of Z to those of V. For instance, by letting z go to ∞ in the identity (3.19), we get the following link between the entrance laws of Z and V:

$$\mathbb{E}_\infty(e^{-xZ_t}) = \mathbb{P}_0(V_t > x) \text{ for } t, x \geq 0. \tag{3.20}$$

Proposition 3.8 (Proposition 3.1 in [F20]). *The Siegmund dual of $(U_t, t \geq 0)$ is the diffusion $(V_t, t \geq 0)$ weak solution to the SDE*

$$\mathrm{d}V_t = \sqrt{cV_t}\mathrm{d}B_t + (c/2 + \Psi(V_t))\mathrm{d}t, \;\; V_0 = y \in (0,\infty), \tag{3.21}$$

where $(B_t, t \geq 0)$ is a Brownian motion[3] *and whose boundary condition at* 0 *and* ∞ *are given in correspondence with that of U in the following way:*

By gathering the correspondences depicted in Tables 3.1 and 3.3, we obtain the following ones between V and Z. Notice that the boundaries 0 of V and ∞ of Z are exchanged but the behaviors of the processes are now symmetric.

3. We stress that processes U and V are meant as weak solutions. The driving Brownian motions, all denoted by B, are not supposed to be the same in the stochastic equations (3.8) and (3.21).

Integral condition	Boundary of U	Boundary of V
$\mathcal{E} = \infty$	0 exit	0 entrance
$\mathcal{E} < \infty$ and $2\lambda/c < 1$	0 regular absorbing	0 regular reflecting
$2\lambda/c \geq 1$	0 entrance	0 exit
$\int^{\infty} \frac{\mathrm{d}x}{\lvert\Psi(x)\rvert} = \infty$	∞ natural	∞ natural
$\int^{\infty} \frac{\mathrm{d}x}{\lvert\Psi(x)\rvert} < \infty$	∞ entrance	∞ exit

Table 3.3 – Boundaries of U, V.

Integral condition	Boundary of V	Boundary of Z
$\mathcal{E} = \infty$	0 entrance	∞ entrance
$\mathcal{E} < \infty$ and $2\lambda/c < 1$	0 regular reflecting	∞ regular reflecting
$2\lambda/c \geq 1$	0 exit	∞ exit
$\int^{\infty} \frac{\mathrm{d}x}{\Psi(x)} = \infty$	∞ natural	0 natural
$\int^{\infty} \frac{\mathrm{d}x}{\Psi(x)} < \infty$	∞ exit	0 exit

Table 3.4 – Boundaries of V, Z.

We characterize now the law of the first explosion time of the LCSBP with the help of V. Let $T_0^{\mathbb{e}_z}$ be the first hitting time of 0 for the bidual process V starting from an independent exponential random variable with parameter z.

Theorem 3.9 (Laplace transform of the first explosion time of LCSBPs, Theorem 3.5 in [F20]). *Assume $\mathcal{E} < \infty$. For all $z \in (0, \infty)$,*

$$\mathbb{E}_z[e^{-\theta\zeta_\infty}] = \int_0^\infty z e^{-zx} \frac{h_\theta^-(x)}{h_\theta^-(0)} \mathrm{d}x = \mathbb{E}[e^{-\theta T_0^{\mathbb{e}_z}}]. \tag{3.22}$$

The function h_θ^- is the unique (up to a multiplicative constant) decreasing solution to the equation

$$\mathscr{G} h = \theta h,$$

where $\mathscr{G}$ is the generator of V:

$$\mathscr{G} f(x) := \frac{c}{2} x f''(x) + \left(\frac{c}{2} + \Psi(x)\right) f'(x). \tag{3.23}$$

Sketch of proof of Theorem 3.9. Recall $\mathscr{L}$ the generator of Z. Let h_θ^- be a decreasing solution to $\mathscr{G}h = \theta h$. Set $f_\theta^+(z) = \int_0^\infty z e^{-xz} h_\theta^-(x)\mathrm{d}x$. Notice first that under the assumption $\mathcal{E} < \infty$, $h_\theta^-(0) < \infty$. This entails that f_θ^+ is a well-defined bounded increasing function. One can check that $\mathscr{L} f_\theta^+ = \theta f_\theta^+$. This provides that the process $(e^{-\theta t} f_\theta^+(Z_t), t \geq 0)$ is a local martingale. The latter is actually a true martingale since f_θ^+ is bounded and by applying the stopping time theorem to ζ_∞^+, we get the first identity in (3.22). The second identity comes from the theory of diffusions, see Annex C.2.2. □

Remark 3.10. A similar study can be done for the extinction time. This will be addressed in Theorem 4.5.

We state now a Laplace duality relationship for the minimal process $Z^{\min}$. This completes the classification of boundaries by adding to Table 3.1 the following line of correspondences:

Integral condition	Boundary of U	Boundary of Z
$\mathcal{E} < \infty$ and $2\lambda/c < 1$	0 regular reflecting	∞ regular absorbing

Theorem 3.11. *Assume $\mathcal{E} < \infty$ and $2\lambda/c < 1$. For any $z \in [0,\infty]$, $x \in [0,\infty]$ and $t \geq 0$*

$$\mathbb{E}_z[e^{-xZ_t^{\min}}] = \mathbb{E}_x[e^{-zU_t^{\mathrm{r}}}], \tag{3.24}$$

with $(U_t^{\mathrm{r}}, t \geq 0)$ the solution to (3.8) with boundary 0 regular reflecting. In particular, for all $z \in (0,\infty)$ and $t \geq 0$,

$$\mathbb{P}_z(\zeta_\infty > t) = \mathbb{E}_0[e^{-zU_t^{\mathrm{r}}}]. \tag{3.25}$$

Remark 3.12. Theorem 3.11 provides also a proof of the explosion of $Z^{\min}$ when $\mathcal{E} < \infty$. Moreover, by letting t go to 0 in (3.25), we see that the boundary ∞ is *regular-for-itself* when it is regular.

We identify now the inverse local time at ∞ of the LCSBP with boundary ∞ regular reflecting. Denote by $(L_t^Z, t \geq 0)$ its local time at ∞ and by $(\tau_x^Z, 0 \leq x < \xi)$ its right-continuous inverse, namely for any $x \geq 0$, $\tau_x^Z := \inf\{t \geq 0 : L_t^Z > x\}$ and $\xi := L_\infty^Z = \inf\{x \geq 0 : \tau_x^Z = \infty\} \in (0,\infty]$. One has

$$\mathcal{I} := \overline{\{t \geq 0 : Z_t = \infty\}} = \overline{\{\tau_x^Z, 0 \leq x < \xi\}} \text{ a.s.}$$

Moreover the process $(\tau_x^Z, x < \xi)$ is a subordinator and ξ its life-time. Note that since ∞ is regular reflecting, the subordinator τ^Z has no drift. Recall also from Proposition 3.8 that 0 is regular reflecting for the bidual process V and call $(L_t^V, t \geq 0)$ its local time at 0.

Theorem 3.13 (Theorem 3.10 in [F20]). *Assume that ∞ is regular reflecting, i.e. $\mathcal{E} < \infty$ and $2\lambda/c < 1$,*

$$(L_t^Z, t \geq 0) \textit{ has the same law as } (L_t^V, t \geq 0).$$

This theorem, perhaps of a more theoretical nature, is established by applying a general result due to Blumenthal and Getoor [29]. In [29, Theorem 1.2], it is shown that local time is characterized by a specific family of excessive functions, defined using the Laplace transform of the first boundary hitting time and the resolvent of the process. This information becomes accessible through Theorem 3.9 and the relation "bidual" (3.19). The latter, in fact, implies that the excessive functions in question, linked to Z and V, are equal.

Numerous properties of local times of diffusions can be applied in principle to the study of κ_Z in order for instance to represent the Lévy measure of τ_Z or its density, see e.g. Borodin and Salminen [32, Chapter II, Section 4]. The latter quantities have no explicit formula when the branching mechanism Ψ is general. However we can identify the packing and Hausdorff dimensions of the zero-set of V. We refer for instance to Bertoin's lecture notes [17, Corollary 9.8]. Since the inverse local times of V and Z coincide in law, we determine in this manner the dimensions of $\mathcal{I}$.

Theorem 3.14 (Theorem 3.12 in [F20]). *Assume $\mathcal{E} < \infty$ and $\frac{2\lambda}{c} < 1$,*

$$\dim_P(\mathcal{I}) = \dim_H(\mathcal{I}) = 2\lambda/c \in [0,1) \textit{ a.s.}$$

By Itô's theory of excursions, since Z and V are Feller processes with boundary ∞ and 0 regular reflecting, their trajectories can be decomposed into excursions out from their boundary ∞ and 0 respectively, see for instance [16, Chapter 4, Section 4].

The process $(e_t, t \leq L^Z_\infty)$ defined by setting for all $t > 0$,

$$e_t = \left(Z_{s+\tau^Z_{t-}}, s \leq \tau^Z_t - \tau^Z_{t-}\right) \text{ if } \tau^Z_t - \tau^Z_{t-} > 0 \text{ and } e_t = \partial \text{ an isolated point, otherwise,}$$

is a Poisson point process on the set of càdlàg excursions out from ∞, stopped at the first infinite excursion, with for σ-finite intensity measure the excursion measure, say, n_Z. We denote an excursion of Z by $\varepsilon : (\varepsilon(t), t \leq \zeta)$ with ζ its length.

Similarly, the diffusion V with 0 regular reflecting has an excursion measure n_V on the set of continuous excursions out of 0. We shall denote an excursion of V by $\omega : (\omega(t), t \leq \ell)$, with ℓ its length. Both boundaries ∞ and 0 being regular reflecting, they are also instantaneous. Since they are moreover regular for themselves, the excursion measures n_Z and n_V are infinite.

The next theorem gathers two results initiating the study of the excursion measure of Z. We first get a relationship between the excursion measures of Z and V, the second provides some information about the law of the infimum of an excursion $I := \inf_{u\in(0,\zeta)} \varepsilon(u)$, under n_Z for LCSBPs that converge towards 0 almost surely.

Theorem 3.15. *Assume ∞ regular reflecting ($\mathcal{E} < \infty$ and $2\lambda/c < 1$).*

— *The following identity holds: for any $x \in [0,\infty)$ and $q > 0$,*

$$n_Z\left(\int_0^\zeta e^{-qu} e^{-x\varepsilon(u)} \mathrm{d}u\right) = n_V\left(\int_0^\ell e^{-qu} \mathbb{1}_{(x,\infty)}(\omega(u)) \mathrm{d}u\right). \tag{3.26}$$

— *If $\Psi(\infty) = \infty$ (i.e. the pure CSBP is not immortal), the law under n_Z of $I := \inf_{0\leq s<\zeta} \varepsilon(s)$, the infimum of an excursion of Z, is given by*

$$n_Z(I \leq a) = 1/S_Z(a),$$

with, for all $a \geq 0$,

$$S_Z(a) := \int_0^\infty \frac{1}{c}\frac{\mathrm{d}x}{x} e^{-ax} e^{-\int_{x_0}^x \frac{2\Psi(u)}{cu}\mathrm{d}u}.$$

3.4 Diffusions on $[0,\infty]$ and Siegmund duality

We provide in this autonomous section a study of Siegmund duality in the framework of diffusions. This is used in Proposition 3.8 in order to identify V the bidual process of Z. This will also be of primary importance for the next chapter.

Siegmund [135, Theorem 1] has established that a standard positive Markov process U whose boundary ∞ is either inaccessible (entrance or natural) or absorbing (exit or regular absorbing) admits a dual process V for $H(u,v) = \mathbb{1}_{\{u<v\}}$ i.e. for all $t \geq 0$, $u, v \in (0,\infty)$,

$$\mathbb{P}_u(U_t < v) = \mathbb{P}_v(V_t > u),$$

if and only if it is *stochastically monotone*, that is to say for any $t \geq 0$ and $y \in (0,\infty)$, the function $x \mapsto \mathbb{P}_x(U_t \leq y)$ is nonincreasing.

Stochastic monotonicity of one-dimensional diffusions is well-known. The study of Siegmund duality in the context of one-dimensional diffusions goes back at least to Cox and Rösler, see [42, Theorem 5]. They first study birth-death processes and then state some facts for diffusions arguing by scaling limits. We refer also the reader to Liggett [112, Chapter II, Section 3], Kolokoltsov [86] and Assiotis et al. [3, Lemma 2.2] for some works on Siegmund duality. Those works however do not treat all boundary cases and we complete them with the following theorem.

Theorem 3.16 (Diffusions and Siegmund duality, Theorem 6.1 in [F20]). *Let $(U_t, t \geq 0)$ be a diffusion on $[0,\infty]$ such that 0 is either inaccessible or absorbing. Assume that its generator is acting on any $f \in C_c^2(0,\infty)$ by*

$$\mathscr{A} f(x) := \frac{1}{2}\sigma^2(x) f''(x) + \mu(x) f'(x), \text{ for all } x \in (0,\infty),$$

with $\sigma^2 \in C^2(0,\infty)$ strictly positive on $(0,\infty)$ and $\mu \in C^1(0,\infty)$.

Then, for any $0 < u, v < \infty$ and any $t \geq 0$

$$\mathbb{P}_u(U_t \geq v) = \mathbb{P}_v(V_t \leq u), \tag{3.27}$$

with $(V_t, t \geq 0)$ a diffusion on $[0,\infty]$ whose generator acts on $C_c^2(0,\infty)$ by

$$\mathscr{G} f(x) := \frac{1}{2}\sigma^2(x) f''(x) + \left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\sigma^2(x) - \mu(x)\right) f'(x). \tag{3.28}$$

In particular, the scale function and speed measure, see their definitions in the Annex, (C.33) and (C.34), are exchanged: up to a multiplicative constant, we have

$$S_U = M_V \text{ and } M_U = S_V,$$

and the following correspondences hold:

Feller's conditions	Boundary of U	Boundary of V
$S_U(0,x] < \infty$ and $M_U(0,x] < \infty$	0 regular	0 regular
$S_U(0,x] = \infty$ and $J_U(0) < \infty$	0 entrance	0 exit
$M_U(0,x] = \infty$ and $I_U(0) < \infty$	0 exit	0 entrance
$I_U(0) = \infty$, $J_U(0) = \infty$	0 natural	0 natural

Table 3.5 – Boundaries of U, V.

*Moreover, when the boundary 0 of both U and V is regular, if one is **absorbing** then necessarily the other is **reflecting**. These correspondences hold too for the boundary ∞ (replacing everywhere 0 by ∞ in Table 3.5).*

The long term behaviors of U and V, when 0 is natural or absorbing for U (namely 0 is either natural, exit or regular absorbing) are also related as follows:

Condition	U	V
$S_U(0,\infty) < \infty$	∞ and 0 attracting	positive recurrence

Table 3.6 – Longterm behaviors of U, V.

Lastly, when ∞ *and* 0 *are attracting for* U*, the stationary law of* V *satisfies*

$$\mathbb{P}(V_\infty \le x) = \mathbb{P}_x(U_t \underset{t\to\infty}{\longrightarrow} \infty) = \frac{S_U(0,x]}{S_U(0,\infty)} \in (0,1) \text{ for any } x \ge 0.$$

Sketch of proof. Theorem 3.16 is established in three steps. We explain them without going into much details for brievety.

Step 1. We check that the dual Markov process V (which exists by Siegmund's theorem) is a Feller process. This can be shown for instance using that the diffusion U is strong Feller and the duality (3.27).

Step 2. We look for the martingale problem solved by the process V until it has left $(0,\infty)$ (minimal process). Let $g \in C^1(0,\infty)$ and $f \in C_c^\infty(0,\infty)$. Set $G(x) = \int_0^x g(u)\mathrm{d}u$ and $F(x) = \int_x^\infty f(t)\mathrm{d}t$. Recall $\mathbb{P}_u(V_s < x) = \mathbb{P}_x(U_s > u)$. Then, integrating this with respect to $f(x)g(u)\mathrm{d}x\mathrm{d}u$ provides

$$\int_0^\infty \mathrm{d}u g(u)\mathbb{E}_u[F(V_s) - F(u)] = \int_0^\infty \mathrm{d}x f(x)\mathbb{E}_x[G(U_s) - G(x)].$$

Since $(U_s, s \ge 0)$ has generator $\mathscr{A}$ then

$$\mathbb{E}_x[G(U_s) - G(x)] = \int_0^s \mathscr{A} P_t^U G(x)\mathrm{d}t,$$

where P_t^U denotes the semigroup of U. By plugging this into the equality above and applying several times Fubini's theorem (about 5 times) as well as several integrations by parts, we end up with

$$\int_0^\infty \mathrm{d}u g(u)\mathbb{E}_u[F(V_s) - F(u) - \int_0^s \mathscr{G} F(V_r)\mathrm{d}r] = 0.$$

The integrand is therefore zero for almost every u. However, due to the Feller property, the integrand is continuous in u, implying that it must be zero for all u. Consequently, this ensures that the process V, up to the exit time of $(0,\infty)$, has generator $\mathscr{G}$.

Step 3. We identify the boundary behavior by the duality relationship (3.27). The fact that $M_U = S_V$ up to a constant follows by direct inspection. □

3.5 Comments

About the explosion

Once the possible behaviors at boundary ∞ delineated, a natural question arises: what can be said about the sample paths as they approach the boundary?

Intuitively, the negative quadratic drift "$-\frac{c}{2}Z_t^2\mathrm{d}t$" pushes down the process out from ∞ and the speed of coming down fluctuates along the curve $t \mapsto 2/ct$. For a thorough study of the behavior of dynamical systems with random perturbations, starting from infinity, we refer the reader to Bansaye [9].

In [F21], with Bo Li and Xiaowen Zhou, we examined the explosion speeds of continuous-state (nonlinear [4]) branching processes. We compare below these speeds to that of coming down from infinity on some examples. We denote the CSBP (*without* competition) by X and ζ_∞ is its explosion time.

4. This is another generalisation of CSBPs in which the individual reproduction rate instead of being constant is a function of the population size

- When the branching mechanism Ψ is regularly varying at 0 with index $\alpha \in (0,1)$, the following renormalisation in law prior to explosion, see Theorem 1.2 in [F21], holds:
$$u_t(0+)X_{\zeta_\infty - t} \underset{t\to 0}{\Longrightarrow} S,$$
where $t \mapsto u_t(0+)$ is the cumulant starting from 0, see Annex A, and, without going into details, S is some positive proper random variable. In this regularly varying setting, it is straightforward to show that the function $u_t(0+)$ is of order $1/t^{\frac{1}{1-\alpha}}$. The latter being much lower than $1/t$, we understand here, at least intuitively, that explosion in a stable branching process will be prevented by the competition, see Example 3.5-1.
- When the branching mechanism Ψ is slowly varying at 0, there is no linear renormalisation of the CSBP prior explosion, but the branching mechanism given in Example 3.5-3 falls into a class we have studied. Let $\beta \in (0,1]$. If X is a CSBP with a Lévy measure such that $\pi(\mathrm{d}u) = \frac{\alpha}{u(\log u)^{\beta+1}}\mathbb{1}_{\{u\geq e\}}\mathrm{d}u$, then its branching mechanism satisfies $\Psi(x) \underset{x\to 0+}{\sim} -\alpha/\log(1/x)^{\beta}$ and we have
$$\log\log X_{\zeta_\infty - t} - \log\log 1/t \underset{t\to 0}{\Longrightarrow} \mathbb{e}_\beta,$$
where $\mathbb{e}_\beta = \mathbb{e}/\beta$ is an exponential random variable with parameter β. We see here that the speed of explosion is at a $\log\log$ scale the same as that of coming down. From that perspective, one can imagine that there is room for some compensation phenomenon to occur near ∞ between the branching and the competition allowing both for explosion and coming down. We actually have shown that for $\beta \in (0,1)$, the boundary is regular. Moreover, the speed of explosion becomes greater than $1/t$ for β close to 0. Heuristically, at the level of the speeds, we see that the battle between the branching and the competition becomes tight. Last, we recall that when $\beta = 1$ there is a phase transition for the explosion according to whether $2\alpha/c > 1$ or not. It seems that we cannot catch it from a simple comparison of the speeds (note that at the $\log\log$ scale neither α nor c really matter).

About the extinction

The problem of conditioning the LCSBP on never getting extinct has been addressed in the work F22 written with Victor Rivero and Anita Winter. Here again the Laplace-Siegmund bidual process V plays a predominant role. It helped us to find an explicit excessive function and to relate our conditioned LCSBP to the process V conditioned on not hitting (or not going to) ∞. We stress that there are several ways to define such conditioning. What we did is to force the total progeny to be infinite and the process obtained along this way is not the Q-process.

Up to my knowledge, whether the Q-process of LCSBPs always exists is not known. It could be interesting to see whether the Q-process (if it exists) associated to the diffusion V absorbed at ∞ can provide information on that of the LCSBP Z at 0. Similar questions could be asked for the quasi-stationary distribution.

Ouverture

- A natural question, already mentioned in the Introduction, is to understand the dualities with U and V at the level of individuals. We could for instance seek a characterization of the two-parameter flow $(Z_t(z), t \geq 0, z \geq 0)$ with the help of the flows related to U and V. However, the latter do not fall into classes of diffusions whose flows properties are well understood.

— In a quite similar fashion, we could look at the Siegmund dual process of the LCSBP. The negative quadratic drift is transformed into a positive one in the dual process, see Theorem 3.16 and causes explosion. From a very intuitive point of view we could imagine that lineages exploding at different times are separating different families from the current time (similarly as what we have seen in Chapter 2 for the subcritical case). However, again this study requires a good understanding of both the flow and its inverse.

CHAPTER 4

First passage times of CSBPs with collision

Summary.

This chapter is devoted to the study of a certain class of processes generalizing the logistic CSBPs by modelling a dynamics of random collisions. In short, additionally to the branching, pair of individuals are picked uniformly in the population, collide and leave a (sub)-critical mass of individuals. Those processes do not satisfy the branching property but as Logistic CSBPs, they satisfy a Laplace duality relationship with some one-dimensional diffusions. Again the world of diffusions being fairly known, this will allow us to obtain a nice panorama on those processes. We will study the first passage times below a level, the attraction to the boundaries and the stationarity distribution. Processes with collision are actually the only Feller processes with no negative jumps to be in Laplace duality with diffusions. This closes somehow the question of which processes can be studied from Laplace duality with diffusions.

We start with a few words on general Markov processes. Let $X := (X_t, t \geq 0)$ be a positive regular[1] Feller process. For $a, b \in \mathbb{R}_+$, we define $\zeta_a^- := \inf\{t \geq 0 : X_t \leq a\}$ and $\zeta_b^+ := \inf\{t \geq 0 : X_t > b\}$, the first passage below level a and above b. Those are stopping times, possibly infinite. The problem of characterizing the laws of those random times, for instance through their Laplace transform, is an entire topic in the theory of stochastic processes. When the process has one-sided jumps, say no negative jumps, general theory, see for instance Cissé et al. [39, Proposition 4.1] for a clear statement, ensures the existence of a family of functions $(f_\theta, \theta > 0)$, such that

$$\mathbb{E}_x\left[e^{-\theta\zeta_a^-}\right] = \frac{f_\theta(x)}{f_\theta(a)}. \tag{4.1}$$

Those functions are decreasing and θ-excessive, i.e. such that $e^{-\theta t}P_t f_\theta \leq f_\theta$ for any $t \geq 0$. They are however only known to be continuous for a certain topology, called the fine topology. The study of ζ_b^+ is even more difficult as it also requires to deal with the possible overshoot. The functions f_θ satisfying (4.1) are explicit or semi-explicit only for a few classes of Markov processes. The most well-known being the Markov processes with continuous sample paths, namely the one-dimensional diffusions, see Annex C. In this framework, the functions f_θ are solutions to certain second order differential equations. For processes with jumps, we mention for instance the spectrally positive Lévy processes, see Kyprianou and Bertoin's books [88, 16], the self-similar processes, see Patie [121]. As already mentioned, see Chapter 1.3, the first entrance times of CSBPs with immigration (CBIs) and extremal shot noise processes (ESNs) have also been studied in [F6] and [F18].

The task of characterizing the law of the first passage times of processes with jumps does not follow from any general treatment unless, by some miracle, the excessive functions are manageable. The CSBPs with collision (I will use the abbreviation CBC) form such a specific class of processes. They have no negative jumps and their first passage times *below a level* can be studied, informally speaking, in the same manner as the diffusions. Background on the latter is given in Annex C.2.2. The main reason lies on the fact that they are *the* Feller processes with no negative jumps in Laplace duality with diffusions.

4.1 CSBPs with collision: first passage times and dualities

4.1.1 Definition through a stochastic equation

We start with a rather cumbersome stochastic equation. One benefit of doing so is perhaps to explain why we use the word of collision. We mention also that this terminology already appears in the literature for some discrete state space processes, see Chen et al. [37].

Consider the following generalisation of the stochastic equation solved by a continuous-state branching process with branching mechanism Ψ for a starting value $z \in [0, \infty)$,

$$\begin{aligned} Z_t = z + \sigma\int_0^t \sqrt{Z_s}\mathrm{d}B_s + b\int_0^t Z_s\mathrm{d}s \\ + \int_0^t\int_0^{Z_{s-}}\int_0^1 h\overline{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u,\mathrm{d}h) + \int_0^t\int_0^{Z_{s-}}\int_1^\infty h\mathcal{N}(\mathrm{d}s,\mathrm{d}u,\mathrm{d}h) \\ + a\int_0^t Z_s\mathrm{d}W_s - \frac{c}{2}\int_0^t Z_s^2\mathrm{d}s + \int_0^t\int_0^{Z_{s-}}\int_0^{Z_{s-}}\int_0^\infty h\overline{\mathcal{M}}(\mathrm{d}s,\mathrm{d}u_1,\mathrm{d}u_2,\mathrm{d}h). \end{aligned} \tag{4.2}$$

1. in the sense that it admits a local time at any point in $(0, \infty)$

— The two first lines in Eq. (4.2) represent the branching dynamics and forms the classical stochastic equation solved by a CSBP, see Annex A, B is a Brownian motion, $\mathcal{N}$ is an independent Poisson random measure on $[0,\infty)^2 \times (0,\infty)$ with intensity $\mathrm{d}s\mathrm{d}u\pi(\mathrm{d}h)$, π is such that $\int_0^\infty 1 \wedge h^2\pi(\mathrm{d}h) < \infty$ and $\overline{\mathcal{N}}$ stands for the compensated random measure, $\overline{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) := \mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) - \mathrm{d}s\mathrm{d}u\pi(\mathrm{d}h)$. Heuristically, prior to an atom of time t of $\mathcal{N}$, an individual u is chosen uniformly in $[0, Z_{t-}]$ and reproduces or dies. The branching part is governed by the Lévy-Khintchine function

$$\Psi(x) := \frac{\sigma^2}{2}x^2 - bx + \int_0^\infty \left(e^{-xh} - 1 + xh\mathbb{1}_{\{h\leq 1\}}\right)\pi(\mathrm{d}h), \quad x \in [0,\infty). \tag{4.3}$$

— The third line in Eq. (4.2) represents *collisions*: again the parameters $a \in [0,\infty)$, $c \in [0,\infty)$ are the diffusive coefficients, W is a Brownian motion, finally $\mathcal{M}$ is an independent Poisson random measure on $[0,\infty)^3 \times (0,\infty)$ with intensity $\mathrm{d}s\mathrm{d}u_1\mathrm{d}u_2\eta(\mathrm{d}h)$, η being a Lévy measure on $(0,\infty)$ satisfying $\int_0^\infty h \wedge h^2\eta(\mathrm{d}h) < \infty$.
Heuristically, prior to an atom of time t of $\mathcal{M}$, two individuals u_1 and u_2 are picked uniformly in the population, they collide and are replaced by an amount h of new individuals "compensated by negative bursts of drift". The collision part is governed by the Lévy-Khintchine function

$$\Sigma(x) := \frac{a^2}{2}x^2 + \frac{c}{2}x + \int_0^\infty \left(e^{-xh} - 1 + xh\right)\eta(\mathrm{d}h), \quad x \in [0,\infty). \tag{4.4}$$

We assume that the collision mechanism Σ is subcritical or critical (i.e. $\Sigma'(0+) = \frac{c}{2} \geq 0$) but not identically zero. Thus collisions are either diminishing the number of individuals or keeping it the same on average. One might thus expect some phenomenon of regulation of the population size when the latter reaches large values. Collisions may for instance prevent or not the growth of the population induced by supercritical branching dynamics. In the case $\Sigma(x) = \frac{c}{2}x$, we recover the logistic CSBP studied in the previous chapter (the generalisation goes thus from a drift to a complete Lévy-Khintchine function of (sub)-critical type).

The existence and uniqueness of the minimal solution of the equation (i.e. up to the hitting times of the boundaries) can be verified by applying a proposition of Palau and Pardo [119], which builds upon the results of Dawson and Li [43] and Fu and Li [64]. The CBCs can also be obtained by time-change techniques. They are linked, in Lamperti's manner, to a class of Markov processes, that generalizes CBIs by substituting the immigration subordinator with a spectrally positive Lévy process. The latter are defined in Vidmar [138]. The stochastic equation (4.2) is merely expository, we will not work with it later.

We call the process Z a CSBP with collision with *branching mechanism* Ψ and *collision mechanism* Σ. To shorten the notation we call it CBC(Ψ, Σ).

Denote by L^Ψ and L^Σ the generators of two Lévy processes with Lévy-Khintchine functions Ψ and Σ. The generator of the CBC Z takes the following form

$$\mathscr{L}f(z) = z\mathrm{L}^\Psi f(z) + z^2\mathrm{L}^\Sigma f(z). \tag{4.5}$$

4.1.2 The role of duality

Laplace duality. An important feature of CBCs lies in the fact that their generator $\mathscr{L}$ satisfies a Laplace duality with the diffusion generator $\mathscr{A}$ given by, for $g \in C^2([0,\infty))$,

$$\mathscr{A}g(x) := \Sigma g''(x) - \Psi(x)g'(x). \tag{4.6}$$

This follows from the following two lines of calculation. Recall the notation $e_x(z) = e_z(x) = e^{-xz}$ and that $\mathrm{L}^{\Psi} e_x(z) = \Psi(x)e_z(x)$ for any mechanism Ψ, see Annex A.2.2 and (A.20). This yields

$$\begin{aligned}\mathscr{L} e_x(z) &= z\mathrm{L}^{\Psi} e_x(z) + z^2\mathrm{L}^{\Sigma} e_x(z)\\ &= \Psi(x) z e_z(x) + \Sigma(x) z^2 e_z(x)\\ &= -\Psi(x)(e_z)'(x) + \Sigma(x)(e_z)''(x)\\ &= \mathscr{A} e_z(x).\end{aligned}$$

When the CBC process $Z := (Z_t, t \geq 0)$ does not explode, the duality can also be established at the level of the semigroups; to wit, for $\{t, x, z\} \subset [0, \infty)$:

$$\mathbb{E}_z\big[e^{-xZ_t}\big] = \mathbb{E}_x\big[e^{-zU_t}\big], \tag{4.7}$$

where U is the diffusion on $[0, \infty)$ with 0 an absorbing state and generator $\mathscr{A}$.

Siegmund duality: Let V be the Siegmund dual of U, recall that it is a Markov process such that

$$\mathbb{P}_x(U_t > y) = \mathbb{P}_y(x > V_t). \tag{4.8}$$

By Theorem 3.16, V is a diffusion with generator

$$\mathscr{G} h := \Sigma h'' + (\Sigma' + \Psi) h'. \tag{4.9}$$

Similarly as in Chapter 3, but with emphasizing on the generators, we summarize the dualities as follows

$$(Z, \mathscr{L}) \overset{\textbf{Laplace dual}}{\longleftrightarrow} (U, \mathscr{A}) \overset{\textbf{Siegmund dual}}{\longleftrightarrow} (V, \mathscr{G}).$$

The duality of generators will in fact suffice for the present study. The following lemma relates analytically the decreasing eigenfunctions of the pseudo-differential operator $\mathscr{L}$ to the increasing ones of the differential operator $\mathscr{G}$. This is the key to establishing all results coming next.

Lemma 4.1 (A decreasing eigenfunction of $\mathscr{L}$, Lemma 5.4 in [F17]). *Let $\theta \in (0, \infty)$, and suppose $h_\theta \in C^2((0, \infty))$ is nonnegative, not zero, nondecreasing and satisfies $\mathscr{G} h_\theta = \theta h_\theta$ on $(0, \infty)$. Put*

$$f_\theta(z) := z\int_0^\infty e^{-zv} h_\theta(v)\mathrm{d}v = h_\theta(0+) + \int_0^\infty e^{-zv} h'_\theta(v)\mathrm{d}v, \quad z \in (0, \infty). \tag{4.10}$$

Then $\mathscr{L} f_\theta = \theta f_\theta$.

Remark 4.2. Notice that Lemma 4.1 does not appeal the operator $\mathscr{A}$ and the diffusion U. The increasing functions h_θ can be related to the latter as follows. For simplifying the discussion, assume that h_θ belongs to the domain[2] of V so that $(e^{-\theta t} h_\theta(V_t), t \geq 0)$ is a martingale. Let μ_θ be the Borel measure on $\mathbb{R}_+$ such that $\mu_\theta([0, x]) := h_\theta(x)$ for all $x \geq 0$, in other words

$$\mu_\theta(\mathrm{d}x) = h_\theta(0+)\delta_0 + h'_\theta(x)\mathrm{d}x.$$

Then, on the one hand, by the Siegmund duality relation (4.8), one sees that μ_θ is a θ-invariant measure for U. On the other hand, by the Laplace duality relationship (4.7), one can check that the Laplace transform of μ_θ is a decreasing θ-invariant function for Z.

2. This is not true in general

We have encountered a similar duality scheme between invariant measures and invariant functions while studying the ancestral lineage process in Chapter 2.5.2, see Theorem 2.20. Note however that in Chapter 2.5, Siegmund duality and Laplace duality were used in the converse order (Siegmund/Laplace instead of Laplace/Siegmund) and only information on the Laplace transform of the invariant measure were available.

4.1.3 First passage times and extinction

We turn to the study of the law of the first passage time of the CBC process Z below a given level. For the sake of simplicity, we work under the assumption that Z does not explode in finite time (i.e. its boundary ∞ is inaccessible). A sufficient condition is that the pure CSBP itself does not explode.

Theorem 4.3 (Theorem 2.8 in [F17]). *Assume that the CBC process does not explode and let $\theta \in (0,\infty)$. There is a unique function $h_\theta \in C^2((0,\infty))$ (up to a multiplicative constant) nonnegative, not zero, nondecreasing solution h on $(0,\infty)$ to*

$$\mathscr{G}h = \Sigma h'' + (\Sigma' + \Psi)h' = \theta h. \tag{4.11}$$

Then, for $\mathsf{a} \leq z$ from (z^*,∞), we have, with f_θ given in (4.10),

$$\mathbb{E}_z\big[e^{-\theta\zeta_{\mathsf{a}}^-}\big] = \frac{f_\theta(z)}{f_\theta(\mathsf{a})}. \tag{4.12}$$

Remark 4.4. When there is no collision, $\Sigma = 0$, and we are not in the subordinator case, the ordinary differential equation (o.d.e.) in (4.11) is of first order and there is a possible singularity at $\varrho := \sup\{x \in [0,\infty) : \Psi(x) = 0\}$, the largest zero of Ψ. Solving the o.d.e. gives for $v \in (\varrho,\infty)$, $h_\theta(v) = e^{\int_{x_0}^{v} \frac{\theta}{\Psi(u)}\mathrm{d}u}$, where (still) $x_0 \in (0,\infty)$ is fixed (and arbitrary). In turn we get

$$f_\theta(z) = z\int_\varrho^\infty e^{-zv} e^{\int_{x_0}^{v} \frac{\theta}{\Psi(u)}\mathrm{d}u}\mathrm{d}v, \quad z \in (0,\infty),$$

and recover then through Formula (4.12) the Laplace transform of the first passage time of the CB(Ψ), see Section 6 in [F6].

Theorem 4.3 deals with first passage times below (accessible) positive levels. Letting a go to 0, we get the law of the extinction time:

Theorem 4.5 (Theorem 2.11 in [F17]). *Assume that the CBC process does not explode. Let $z \in (z^*,\infty)$. The following equivalence holds true.*

$$\mathbb{P}_z(\zeta_0^- < \infty) > 0 \text{ if and only if } \int^\infty \frac{\mathrm{d}u}{|\Psi(u)|} < \infty. \tag{4.13}$$

We recall that if the integral converges then necessarily $\Psi(\infty) = +\infty$, i.e. $-\Psi$ is not the Laplace exponent of a subordinator, see Annex A.2.5. Furthermore, the Laplace transform of the extinction time of Z satisfies:

$$\mathbb{E}_z\big[e^{-\theta\zeta_0^-}\big] = \int_0^\infty z e^{-zx}\frac{h_\theta(x)}{h_\theta(\infty)}\mathrm{d}x = \mathbb{E}\big[e^{-\theta T_\infty^{\mathbb{e}_z}}\big], \quad \theta \in (0,\infty), \tag{4.14}$$

where $T_\infty^{\mathbb{e}_z}$ denotes the explosion time of the bidual minimal[3] *diffusion V when the latter starts from an independent exponential random variable $\mathbb{e}_z$ with rate z.*

We see in (4.13) that collisions are never causing extinction in finite time, since we recover the Grey's condition for extinction of the CSBP(Ψ). Note also that (4.14) is reminiscent to (3.22).

3. namely with boundaries 0 and ∞ absorbing if they are accessible

4.1.4 Classification of attracting boundaries

It is well-known that the long-term behaviors of a one-dimensional diffusion is characterized through its speed measure and scale function, see Annex C.

Similarly, we can use them to classify those of Z. *Throughout the remainder of this chapter we fix an arbitrary $x_0 \in (0,\infty)$.* Then set

$$S_V(x) := \int_{x_0}^{x} \frac{1}{\Sigma(u)} e^{\int_u^{x_0} \frac{\Psi(v)}{\Sigma(v)} \mathrm{d}v} \mathrm{d}u, \quad x \in (0,\infty), \tag{4.15}$$

for the scale function of V. By abuse of notation denote by S_V also its associated Lebesgue-Stieltjes measure on $(0,\infty)$; to wit, for $\mathsf{a} < \mathsf{b}$ from $(0,\infty)$,

$$S_V(\mathsf{a},\mathsf{b}] = S_V(\mathsf{b}) - S_V(\mathsf{a}) = \int_{\mathsf{a}}^{\mathsf{b}} \frac{1}{\Sigma(x)} e^{\int_x^{x_0} \frac{\Psi(u)}{\Sigma(u)} \mathrm{d}u} \mathrm{d}x \in (0,\infty), \tag{4.16}$$

which determines S_V uniquely. Finally, introduce

$$S_Z(w) := \int_0^\infty e^{-xw} S_V(\mathrm{d}x) = \int_0^\infty \frac{e^{-xw}}{\Sigma(x)} e^{\int_x^{x_0} \frac{\Psi(u)}{\Sigma(u)} \mathrm{d}u} \mathrm{d}x, \quad w \in (0,\infty). \tag{4.17}$$

Our next theorem provides necessary and sufficient conditions for the boundaries 0 and ∞ to be attracting, by which we mean that the process tends towards the boundary with positive probability. These conditions are those of the diffusion V for the boundaries ∞ and 0, respectively. At the risk of boring a bit the reader, I give them below.

Set

$$z^\star = \left(\limsup_{u\to\infty} -\Psi(u)/\Sigma(u)\right) \vee 0. \tag{4.18}$$

When $z^\star > 0$, the process Z stays above this value after it has crossed it.

Theorem 4.6 (Attracting boundaries, Theorem 2.3 in [F17]). *Let $z^\star < \mathsf{a} < z < \infty$.*

1. *If $S_V(0,x_0] = \infty$ then $\mathbb{P}_z(\zeta_{\mathsf{a}}^- < \zeta_\infty) = 1$.*
2. *If $S_V(0,x_0] < \infty$ then $\mathbb{P}_z(\zeta_{\mathsf{a}}^- < \zeta_\infty) = \frac{S_Z(z)}{S_Z(\mathsf{a})} \in (0,1)$.*
3. *$Z_t \underset{t\to\infty}{\longrightarrow} \infty$ with positive $\mathbb{P}_z$-probability (respectively, $\mathbb{P}_z$-almost surely) if and only if $S_V(0,x_0] < \infty$ (respectively, $S_V(0,x_0] < \infty$ and $S_V(x_0,\infty) = \infty$).*
4. *Suppose $\Psi \neq 0$. Then $Z_t \underset{t\to\infty}{\longrightarrow} 0$ with positive $\mathbb{P}_z$-probability (respectively, $\mathbb{P}_z$-almost surely) if and only if $S_V(x_0,\infty) < \infty$ (respectively, $S_V(x_0,\infty) < \infty$ and $S_V(0,x_0] = \infty$). When $S_V(0,\infty) < \infty$, then, moreover, $\mathbb{P}_z(Z_t \underset{t\to\infty}{\longrightarrow} 0) = 1 - \mathbb{P}_z(Z_t \underset{t\to\infty}{\longrightarrow} \infty) = \frac{S_Z(z)}{S_Z(0)} \in (0,1)$.*
5. *If $\Psi = 0$ then $\mathbb{P}_z$-almost surely $Z_t \underset{t\to\infty}{\longrightarrow} 0$.*

Theorem 4.6 actually states the following correspondences:

Condition	Boundary of Z	Boundary of V
$S_V(0,x_0] < \infty$	∞ attracting	0 attracting
$\Psi = 0$ or $S_V(x_0,\infty) < \infty$	0 attracting	∞ attracting

Table 4.1 – Attracting boundaries of Z and V

Remark 4.7. The convergence towards ∞ in Theorem 4.6(3), when $S_V(0,x_0] < \infty$, hides two different possibilities: the CBC process can be either transient (∞ is attracting, but not accessible) or can explode (∞ is accessible). Indeed the condition $S_V(0,x_0] = \infty$ is not necessary in general for the process to be non-explosive, see Example 4.8(1) below. In the case $\Sigma(x) = \frac{c}{2}x$, $x \in [0,\infty)$, however, the condition $S_V(0,x_0] = \infty$ turns out to be equivalent to $\mathcal{E} = \infty$, see Chapter 3.1. No transience phenomenon can occur in logistic CBs, they can only converge to ∞ by reaching it.

In the non-subordinator case, i.e. $\Psi(\infty) = \infty$, one can easily check that $S_V(x_0,\infty) < \infty$ always holds. So, by Theorem 4.6(4), the necessary and sufficient condition for almost sure convergence towards 0 is then $S_V(0,x_0] = \infty$. In the subordinator case, the condition $S_V(x_0,\infty) < \infty$ may or may not be satisfied. In other words, collisions can be strong enough ($S_V(x_0,\infty) < \infty$) or not ($S_V(x_0,\infty) = \infty$) for the event of convergence towards 0 to have positive probability or not. Lastly, in the (sub)critical branching case, one always has $S_V(0,x_0] = \infty$.

Example 4.8 (Example 2.5 in [F17]).

1. Let $a > 0$ and $b \in \mathbb{R}$. One of the simplest CBCs is the process with mechanisms
$$\Sigma(x) = \tfrac{a^2}{2}x^2 \text{ and } \Psi(x) = -bx,\ x \in [0,\infty).$$
It satisfies the SDE
$$\mathrm{d}Z_t = aZ_t\mathrm{d}W_t + bZ_t\mathrm{d}t,\ Z_0 = z,$$
which corresponds to a geometric Brownian motion, namely for all $t \geq 0$,
$$Z_t = z\exp\left((b - \frac{a^2}{2})t + aW_t\right).$$
One can directly check that $S_V(0,x_0] < \infty$ if and only if $b > \frac{a^2}{2}$, in which case the process $(Z_t, t \geq 0)$ is transient (and does not explode). We also see that Brownian collisions regulate the deterministic growth, that is to say, $Z_t \underset{t\to\infty}{\longrightarrow} 0$ a.s., when $\frac{a^2}{2} > b$.
2. More generally if $\Psi'(0+) =: -b \in \mathbb{R}$ and $\Sigma(x) \underset{x\to 0}{\sim} \frac{a}{2}x^2$, then $S_V(0,x_0] = \infty$ if and only if $b \leq \frac{a^2}{2}$. If in addition to the latter condition $\Psi(x) > 0$ for some $x > 0$, then $S_V(x_0,\infty) < \infty$ and thus $Z_t \underset{t\to\infty}{\longrightarrow} 0$ a.s..
3. Consider $\Sigma(x) = dx^\alpha$ with $\alpha \in (1,2)$ and $\Psi(x) = -d'x^\beta =: -\Phi(x)$ with $\beta \in (0,1)$, $x \in [0,\infty)$. Then we have as follows.
   - If $\beta > \alpha - 1$, neither 0 nor ∞ is attracting.
   - If $\beta < \alpha - 1$, 0 and ∞ are both attracting.
   - If $\beta = \alpha - 1$, ∞ is attracting if and only if $d'/d > \alpha - 1$, while 0 is attracting if and only if $d'/d < \alpha - 1$. In the case of equality, $d'/d = \alpha - 1$, neither 0 nor ∞ are attracting.
4. Finally, consider the case when $\Sigma(x) = dx^\alpha$ for all $x \in [0,\infty)$, with $\alpha \in (1,2)$, and a branching mechanism Ψ such that $\Psi'(0+) \in (-\infty,\infty)$. Then 0 is attracting, and, if moreover $\Psi(x) \geq 0$ for some $x > 0$, then $Z_t \underset{t\to\infty}{\longrightarrow} 0$ a.s..

4.1.5 Stationary distribution

As observed in Example 4.8(3), in the subordinator case, some phenomenon of recurrence can occur and a stationary regime may exist. Let M_V be the speed measure of V on $(0,\infty)$: for $\mathsf{a} < \mathsf{b}$ from $(0,\infty)$,

$$M_V(\mathsf{a},\mathsf{b}] = \int_{\mathsf{a}}^{\mathsf{b}} e^{\int_{x_0}^{x} \frac{\Psi(u)}{\Sigma(u)}\mathrm{d}u}\mathrm{d}x \in (0,\infty), \tag{4.19}$$

where, still, $x_0 \in (0,\infty)$ is arbitrary but fixed.

Theorem 4.9 (Stationary distribution and long-term behavior, Theorem 2.14 in [F17]). *Assume that $S_V(0,x_0] = \infty$ and $S_V(x_0,\infty) = \infty$. Let $z \in (0,\infty)$. Then the CBC process converges in law towards a non-degenerate random variable Z_∞ on (z^*,∞) if and only if $M_V(0,\infty) < \infty$. Moreover, the Laplace transform of the latter is then given by*

$$\mathbb{E}_z[e^{-xZ_\infty}] = \frac{M_V(x,\infty)}{M_V(0,\infty)}, \quad x \in [0,\infty). \tag{4.20}$$

The case $M_V(0,\infty) = \infty$ covers three different possibilities:

1. *If $M_V(0,x_0] < \infty$ and $M_V(x_0,\infty) = \infty$, then $Z_t \underset{t\to\infty}{\longrightarrow} 0$ in probability.*
2. *If $M_V(0,x_0] = \infty$ and $M_V(x_0,\infty) < \infty$, then $Z_t \underset{t\to\infty}{\longrightarrow} \infty$ in probability.*
3. *If $M_V(0,x_0] = \infty$ and $M_V(x_0,\infty) = \infty$, then Z has no limiting distribution.*

Remark 4.10. Plainly, if $-\Psi$ is not the Laplace exponent of a subordinator, then $M_V(x_0,\infty) = \infty$. Here are two simple conditions ensuring, between them, that $M_V(0,\infty) < \infty$ and hence that a limiting distribution exists. If $\Sigma'(0+) = c/2 > 0$, then $M_V(0,x_0] < \infty$ (without further assumptions on Ψ). If $-\Psi$ is the Laplace exponent of a subordinator with drift d, i.e. $-\Psi(x)/x \underset{x\to\infty}{\longrightarrow} \mathrm{d} > 0$, such that $\frac{2\mathrm{d}}{a^2} > 1$ (with $a \geq 0$ the diffusive coefficient in (4.4) and by convention $1/0 = \infty$), then $M_V(x_0,\infty) < \infty$.

Remark 4.11. One verifies easily from (4.20) that the limiting distribution of the CBC admits a first moment if and only if $\int_0^{x_0} \frac{-\Psi(u)}{\Sigma(u)}\mathrm{d}u < \infty$.

Remark 4.12. Theorem 4.9 generalizes Theorem 3.6 for which $\Sigma(x) = \frac{c}{2}x$. The condition **A** matches with $M_V(0,\infty) < \infty$.

Example 4.13.

1. Consider the CBC process with collision and branching mechanisms satisfying, for $x \in [0,\infty)$, $\Sigma(x) = \frac{a^2}{2}x^2 + \frac{c}{2}x$ with $a, c \in (0,\infty)$ and $\Psi(x) = -\mu x$ with $\mu \in \mathbb{R}$. In other words, $(Z_t, t \geq 0)$ satisfies the SDE, sometimes called stochastic Verhulst equation
$$\mathrm{d}Z_t = aZ_t\mathrm{d}W_t + (\mu Z_t - \frac{c}{2}Z_t^2)\mathrm{d}t, \quad Z_0 = z.$$
See Giet et al. [66] for a deep study of this diffusion (including its first passage times). The CBC process Z admits a limiting distribution if and only if $\mu > \frac{a^2}{2}$. When it exists, the latter has for its Laplace transform
$$\mathbb{E}[e^{-xZ_\infty}] = \left(\frac{a^2}{c}x + 1\right)^{-(\frac{2\mu}{a^2}-1)}, \quad x \in [0,\infty),$$
which is the Laplace transform of a gamma distribution with density
$$(0,\infty) \ni u \mapsto \frac{\beta^\alpha}{\Gamma(\alpha)}u^{\alpha-1}e^{-\beta u},$$
its parameters being $\alpha := \frac{2\mu}{a^2} - 1$ and $\beta := \frac{c}{a^2}$.

2. Assume that, for $x \in [0,\infty)$, $\Sigma(x) = dx^\alpha$ with $\alpha \in (1,2)$ and $\Psi(x) = -d'x^\beta$ with $\beta > \alpha - 1$ and $d, d' \in (0,\infty)$. Then the CBC process Z admits a limiting distribution with Laplace transform:
$$\mathbb{E}[e^{-xZ_\infty}] = \frac{\int_x^\infty e^{-\frac{d'}{d}u^{\beta-\alpha+1}}\mathrm{d}u}{\int_0^\infty e^{-\frac{d'}{d}u^{\beta-\alpha+1}}\mathrm{d}u} = \frac{\Gamma\left(\frac{1}{\beta-\alpha+1}, \frac{d'}{d}x^{\beta-\alpha+1}\right)}{\Gamma\left(\frac{1}{\beta-\alpha+1}\right)}, \quad x \in [0,\infty),$$
where $\Gamma(s,x) := \int_x^\infty u^{s-1}e^{-u}\mathrm{d}u$ (for $s > 0$ and $x \geq 0$) is the incomplete Gamma function.
3. Assume that, for $x \in [0,\infty)$, $\Sigma(x) = dx^\alpha$ with $\alpha \in (1,2)$, $d \in (0,\infty)$ and $\Psi(x) = -dx^{\alpha-1}$. Then if $d'/d < 1$, $M_V(x_0,\infty) = \infty$ and $M_V(0,x_0] < \infty$, thus Z tends to 0 in probability. If $d'/d > 1$, $M_V(x_0,\infty) < \infty$ and $M_V(0,x_0] = \infty$, and again Z tends to ∞ in probability. In the case $d'/d = 1$, Z has no limiting distribution.

4.2 Characterization of CSBPs with collision

The final substantial result on which we report here establishes that Laplace duality with a diffusion at the level of the generators actually characterizes CBCs. In order to formulate this with ease we suspend temporarily all meaning attached hitherto to Z, $\mathscr{L}$, $\mathscr{A}$, Σ and Ψ.

Theorem 4.14 (Theorem 2.21 in [F17]). *Let $\mathscr{L}$ be the infinitesimal generator of a positive (possibly explosive) Feller process $(Z_t, t \geq 0)$ with no negative jumps and with 0 an absorbing state. Assume that its domain includes the set of the exponential functions $\{e_x(\cdot), x > 0\}$. Suppose further that $\mathscr{L}$ is in Laplace duality with the **conservative** generator of a diffusion process on $[0,\infty)$, more precisely, suppose that*
$$\mathscr{L}e_x(z) = \Sigma(x)z^2e^{-xz} + \Psi(x)ze^{-xz} =: \mathscr{A}_x e^{-xz}, \quad \{x,z\} \subset [0,\infty), \tag{4.21}$$
holds true for some $\Sigma : [0,\infty) \to [0,\infty)$, not zero, and some $\Psi : [0,\infty) \to \mathbb{R}$, both continuous at zero. Then Ψ and Σ are Lévy-Khintchine functions of the spectrally positive type as in (4.3)-(4.4) and $\mathscr{L}$ acts on $C_c^\infty([0,\infty))$ according to (4.5).

4.3 Comments

Literature and a remaining question

We have not addressed entirely the classification of the boundary ∞ of the CSBPs with collision. We expect that it can be again regular as in the LCSBPs and that the same correspondences would hold. When the collision mechanism is critical (i.e. there is no logistic part), Feller's tests for the diffusion V do not simplify and many behaviors at ∞ are possible, including the case of a natural attracting boundary.
It is natural to wonder what happen if the collisions are supercritical, i.e. $\Sigma'(0) \in [-\infty, 0)$. In this setting, a term of positive quadratic drift emerges and the process explodes with positive probability. Additionally, note that the duality (4.21) is destroyed since Σ is negative near 0 and thus cannot be a diffusive coefficient.

The CBC process with collision mechanism $\Sigma(x) = \frac{cx}{2} + a^2\frac{x^2}{2}$ can be seen as a continuous-state branching process in a Brownian random environment, we refer to Leman and Pardo [103], see also He et al. [74] and Palau and Pardo [119] for the case of a general Lévy random environment.

Ouvertures

— Theorem 4.14 naturally raises the question of whether other Markov processes (or semigroups and infinitesimal generators) beyond those studied here satisfy a Laplace duality relationship, and if so, whether they can be completely characterized (as Siegmund did for the indicator function at the level of processes, see [135]). This is an ongoing project. I wish to highlight that starting from the Courrège-Von Wandelfeds form of a generator, we can taylor-make a *very* large class of generators satisfying Laplace duality. This includes, for instance, the branching processes with immigration and in Lévy random environments.

— The flow of CSBPs with collision indexed by the initial values has not been studied. It is noteworthy that the two stochastic integrals with respect to the independent Brownian motions in (4.2) can be rewritten respectively as follows:

$$\int_0^t \sqrt{Z_s}\mathrm{d}B_s = \int_0^t \int_{[0,Z_s]} \widetilde{B}(\mathrm{d}s, \mathrm{d}u) \text{ and } \int_0^t Z_s \mathrm{d}W_s = \int_0^t \int_{[0,Z_s]\times[0,Z_s]} \widetilde{W}(\mathrm{d}s, \mathrm{d}u_1, \mathrm{d}u_2),$$

with $\widetilde{B}(\mathrm{d}s, \mathrm{d}u)$ and $\widetilde{W}(\mathrm{d}s, \mathrm{d}u_1, \mathrm{d}u_2)$ independent Gaussian time-space white noises on $(0, \infty) \times (0, \infty)$ and $(0, \infty) \times (0, \infty)^2$ based on the Lebesgue measures $\mathrm{d}s\mathrm{d}u$ and $\mathrm{d}s\mathrm{d}u_1\mathrm{d}u_2$, respectively. This allows one to interpret also both diffusive parts in terms of branching and collision. We refer to the Annex A for references on white noises. In the same spirit as for the flow of CSBPs which can be constructed from a Poisson point process on the càdlàg paths space, see Chapter 1, one may wonder if it is possible to give a Poisson construction of the flow of CSBPs with collision. Of course, since the latter does not have the branching property, if such representation exists, it will not be given by a simple summation of independent paths.

— Finally, a potential other direction would be to go to the two-dimensional setting and define a process with collision between types. Such processes would be reminiscent to the catalytic branching processes, see e.g. [110] and the references therein.

Part III

Exchangeable Fragmentation-Coalescence and Moment duality

In this last part, we study Exchangeable Fragmentation-Coalescence (EFC) processes and some Markov processes valued in $[0,1]$ with jumps, called Λ-Wright-Fisher processes with selection.

The EFCs are processes valued in the set of integers partitions and can be seen as a generalisation of exchangeable coalescents. Additionally to the dynamics of merging, blocks can fragmentate into sub-blocks. Those processes have been defined by Julien Berestycki [11], at about the same time as the logistic CSBP was introduced by Amaury Lambert. Contrary to the consecutive coalescent processes, seen in Chapter 2, and in which blocks were simply consecutive intervals with i.i.d. lengths, here the main assumption on the random partition is its *exchangeability*. That is to say, at any given time, the law of the partition remains the same if one permutes the integers. For the sake of conciseness, the notion of exchangeability and the structure it induced (as for instance Kingman's paint-box representation) will not be explained in details.

We shall focus on the setting in which a fragmentation only dislocates one block at a time and no simultaneous multiple coagulations occur (as in a Λ-coalescent). This class of Markov processes might be seen at a first glance as very different from the processes we studied before, however, once we look at the functional of the number of blocks, we end up with a process valued in $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, which can also be interpreted as a continuous-time branching process with interactions [4]. The branching is immortal (i.e. with no natural death) and driven by fragmentation and the interactions are governed by the coalescences.

In Chapter 5, based on [F12] and [F13], we investigate the nature of the boundary ∞ for their block-counting process. We focus on the setting where the coalescent part, solely would make the number of blocks, come down from infinity (i.e. ∞ is an entrance boundary). Our main assumption on the fragmentation is that it occurs at a finite rate. A block however can be dislocated into an arbitrary number of sub-blocks, possibly infinite. We shall address the three following questions: If we start from infinitely many blocks, is the coalescence strong enough for having finitely many ones at some time? In such a case, we would say that the EFC process comes down from infinity (here ∞ is either an entrance or a regular boundary, see Annex C.1). In the other way round, if we start from a finite number of blocks, is the fragmentation strong enough for having infinitely many ones at some time? We would say that the EFC process explodes (∞ is accessible). Last but not least, can we find regimes where the configuration with infinitely many blocks is regular, and if so is it regular reflecting, regular for itself?

In Chapter 6, extirped from [F16], the last of this document, we look at a class of Markov processes valued in $[0,1]$, called Λ-Wright-Fisher processes with selection. Those processes occur for instance for modelling the frequency of a disadvantaged allele in a two-allele model driven by a *genetic drift* [5] and a selection mechanism. A typical question in theoretical genetics is to see whether allelic diversity is maintained or if at the contrary there is *fixation* in finite time of one of the two alleles. Namely, the process is absorbed in finite time at 0 or 1. In contrast to CSBPs, Λ-Wright-Fisher processes have both positive and negative jumps. They also bear no direct relation to Lévy processes, alike the CSBPs through the Lamperti time-change (see Annex A.2.3). This makes their study more complex. They lie however in *moment* duality, i.e. in H-duality, with $H(x,n) = x^n$, with the block-counting process of a Λ-coalescent.

4. The primary motivation for the study of EFC however was not so much coming from this point of view, but from that of partition-valued processes.

5. which we recall is not a drift in the usual mathematical sense but the randomness in allele's resampling between generations

When the selection term takes a specific form, such a duality relationship also holds with the block-counting processes of an EFC process. In the same fashion as in Part II with Laplace duality, by using the results of the previous chapter on the EFCs, we will be able to connect many properties at the boundaries for the two classes. We stress however that neither of the two processes is significantly simpler to study. By alternating between the two processes, and focusing on the one that is easier to study for a given question, we will provide a comprehensive picture of what happens at the boundaries for both processes. In particular, in a similar spirit as the extension defined past explosion of the LCSBPs, see Chapter 3, we will define an extension of the Λ-Wright-Fisher process beyond the fixation of the disadvantageous allele. In other words, even if the genetic drift is strong enough to cause fixation of the deleterious allele, certain forms of selection will allow the population to start (or restart) from a configuration where all genes are carrying the disadvantageous allele.

CHAPTER 5

On the number of blocks in EFC processes

Summary.

We study in this chapter the functional of the number of blocks (also called block-counting process) of simple EFC processes. This is a process valued in $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, namely the one-point compactification of the set of integers. Assuming Schweinsberg's condition for coming down from infinity of the coalescent part, we find two parameters $\theta_\star \leq \theta^\star$, valued in $[0, \infty]$, that measure somehow the combination of both coagulation and fragmentation strengths. When both parameters agree (typically in regularly varying settings), we call their common value θ, and a phase transition occurs at $\theta = 1$: if $\theta < 1$ the EFC comes down from infinity and if $\theta > 1$, it stays infinite. We recover in particular a phase transition, similar to that obtained by Kyprianou, Pagett, Rogers, Schweinsberg, see [89] for the so-called "Fast" fragmentation-coalescence.

When the fragmentation dislocates blocks into finitely many sub-blocks, the question whether explosion will occur or not is more subtle. One can find easily a Lyapunov function ensuring the non-explosion. For the explosion however, we shall need to establish a new sufficient Lyapunov-type condition for explosion of an integer-valued continuous-time Markov chain with both-sided jumps.
Applying those results allows us to get different regimes for the boundary ∞ to be entrance or exit. We stress also on the fact that the condition for explosion and that for coming down from infinity do not exclude each others. We shall see indeed that when the fragmentation and coagulation measures are regularly varying, the boundary ∞ can be an entrance, an exit or a regular boundary. In the latter setting, the EFC process leaves and returns to partitions with infinitely many blocks.

5.1 Simple EFC processes and the block-counting process

Exchangeable fragmentation-coalescence (EFC) processes are Feller processes with state-space $\mathcal{P}_\infty$, the space of partitions of $\mathbb{N}$, that are evolving by mergings of equivalence classes or fragmentation of one class into sub-classes. They have been first introduced by J. Berestycki in [11].

We collect below some facts on partitions and EFCs. We shall mainly be interested in the number of equivalence classes (called blocks or fragments) in the process, so that the reader whose primary interest is not the partition setup, can skip this part, and go directly to Lemma 5.2. The fact however that we work with a process valued in the set of partitions guarantees that the process under study is well-defined when it is starting from ∞, in particular then after explosion if the latter occurs.

Any partition $\pi \in \mathcal{P}_\infty$ is seen as the list of the equivalence classes (called blocks) ordered by their least element: $\pi = (\pi_1, \pi_2, \cdots)$ and if $i \leq j$, then $\min \pi_i \leq \min \pi_j$. For any partition π, we denote by $\#\pi$ the number of the non-empty blocks. We take the convention that $\pi_i = \emptyset$ for any $i > \#\pi$. The space $\mathcal{P}_\infty$ is endowed with the compact metric $d(\pi, \pi') := (\max\{n \geq 1 : \pi_{|[n]} = \pi'_{|[n]}\})^{-1}$ where $\pi_{|[n]} = (\pi_1 \cap [n], \pi_2 \cap [n] \cdots)$ is the restriction to $[n] := \{1, \cdots, n\}$ of π. The set of partitions of $[n]$ is denoted by $\mathcal{P}_n$.

Definition 1.A. *Let $n \in \mathbb{N} \cup \{\infty\}$ and $m \in \mathbb{N} \cup \{\infty\}$, $\pi \in \mathcal{P}_n$ and $\pi' \in \mathcal{P}_m$ and $k \in \mathbb{N}$.*

— *If $\#\pi \leq m$, a coagulation of π by π', denoted by $\mathrm{Coag}(\pi, \pi')$, is a partition of $[n]$ defined by*
$$\mathrm{Coag}(\pi, \pi') := \{\underset{j \in \pi'_i}{\cup} \pi_j \ ; \ i \geq 1\}.$$

— *If $\#\pi \geq \ell$, a fragmentation of the ℓ-th block of π by π', denoted by $\mathrm{Frag}(\pi, \pi', \ell)$, is the collection of sets*
$$\mathrm{Frag}(\pi, \pi', \ell) := \Big(\{\pi_i \ ; \ i \in [|1, \#\pi|] \setminus \{\ell\}\} \cup \{\pi_\ell \cap \pi'_j \ , j \geq 1\}\Big)^{\downarrow}$$
where the notation $(\dots)^{\downarrow}$ means that we are reindexing by their least element the collection of sets formed by the sub-blocks of π_k according to π' and all π_i for $i \neq \ell$.

For instance, let $\pi = \{\{1, 3\}, \{2, 5\}, \{4\}\}$, $\pi' = \{\{1\}, \{2, 3\}\}$ and $\ell = 1$. Then,
$$\begin{aligned}\mathrm{Coag}(\pi, \pi') &= \{\{1, 3\}, \{2, 4, 5\}\}, \text{ and}\\ \mathrm{Frag}(\pi, \pi', 1) &= \{\{1, 3\} \cap \{1\}, \{1, 3\} \cap \{2, 3\}, \{2, 5\}, \{4\}\}^{\downarrow}\\ &= \{\{1\}, \{2, 5\}, \{3\}, \{4\}\}.\end{aligned}$$

For any $\pi \in \mathcal{P}_n, \pi' \in \mathcal{P}_m$ with $m \geq \#\pi$, the partition $\mathrm{Coag}(\pi, \pi')$ is coarser than π. For any $\ell \leq \#\pi$, when $m \geq \max \pi_\ell$, $\pi_\ell \cap [m] = \pi_\ell$ and $\cup_{i \geq 1} \mathrm{Frag}(\pi, \pi', \ell)_i = \cup_{\substack{i=1\\ i \neq \ell}}^{\#\pi} \pi_i \cup (\pi_\ell \cap [m]) = [n]$, so that $\mathrm{Frag}(\pi, \pi', \ell)$ is also a partition of $[n]$, which is finer than π.

Definition 1.B. *An EFC process is a Markov process $(\Pi(t), t \geq 0)$ valued in $\mathcal{P}_\infty$ such that its restrictions to any finite interval $[n]$ evolve as follows:*
$$\Pi_{|[n]}(t) = \mathrm{Coag}(\Pi_{|[n]}(t-), \pi_c) \text{ or } \Pi_{|[n]}(t) = \mathrm{Frag}(\Pi_{|[n]}(t-), \pi_f, \ell)$$
with π_c is an exchangeable partition, π_f is an exchangeable partition and ℓ belongs to $[n-1]$.

*We say that Π is a **simple** EFC is π_c has only one non-singleton block, the total rate of fragmentation is finite and there are no fragmentations into singletons.*

Remark 5.1. There are no multiple simultaneous coalescences in a simple EFC. The name coined “simple” follows Bertoin’s terminology for the Λ-coalescents.

Among other things, it is established in [11] that any EFC is characterized in law by two σ-finite exchangeable measures on $\mathcal{P}_\infty$, μ_{Coag} the measure of coagulation and μ_{Frag}, that of fragmentation. Any EFC process can be constructed from independent Poisson point processes valued respectively in $\mathcal{P}_\infty$ and $\mathcal{P}_\infty \times \mathbb{N}$ with intensity μ_{Coag} and $\mu_{\text{Frag}} \otimes \#$ (the hashtag denotes the counting measure). We do not enter into the details and give below the minimal information on them that we need for presenting the results. We refer the reader to [11] for the general form that those measures can take, as well as the integrability conditions they must satisfy.

We denote the block-counting process by $\#\Pi$. Notice that it has an infinite life-time. There are several pitfalls to circumvent in order to study $\#\Pi$. To mention some, notice that the counting function $\# : \pi \mapsto \#\pi$ is not continuous on $\mathcal{P}_\infty$: for instance, consider the sequence (π^k) with $\pi^k := ([k], \{k+1\}, \cdots)$ whose only non-singleton block is the first. One has $d(\pi^k, 1_{\mathbb{N}})$ goes to 0 as k goes to ∞, with $1_{\mathbb{N}} := (\mathbb{N}, \emptyset, \cdots)$, even though $\#\pi^k = \infty$ for all k. Another important drawback compared to the pure coalescence framework is that the process of counting the blocks of the restriction, $\#\Pi_{|[n]}$, is not Markovian, since the position after a fragmentation depends not only on the number of blocks before the jump, but also on the shape of the blocks (think of the case where $\Pi_{|[n]}$ contains a singleton, which cannot be dislocated). To avoid these kinds of difficulty, we assume that there are no fragmentations into singletons, this guarantees that there is no formation of dust (i.e. singletons) in a simple EFC process and simplifies a bit the study.

The exchangeability property entails that the rates at which respectively k given blocks among n merge and k blocks are created after a fragmentation in a simple EFC are of the following form:

— For any $2 \leq k \leq n$,

$$\begin{aligned}\lambda_{n,k} &:= \mu_{\text{Coag}}(\{\pi \in \mathcal{P}_\infty : \text{ the non-singleton block of } \pi_{|[n]} \text{ has } k \text{ elements }\}) \\ &= \int_{(0,1]} x^k (1-x)^{n-k} x^{-2} \Lambda(\mathrm{d}x) + \Lambda(\{0\}) \mathbb{1}_{\{k=2\}},\end{aligned} \tag{5.1}$$

for some finite measure Λ on $[0,1]$. The coalescences are governed as in a Λ-coalescent.

— For any $k \geq 2$,

$$\mu(k) := \mu_{\text{Frag}}(\{\pi \in \mathcal{P}_\infty : \#\pi = k+1\}). \tag{5.2}$$

We *always* assume from now on that Λ gives no mass at 1 so that not all blocks can merge at once. The binary mergings (the Kingman part) have rate $c_{\text{k}} := \Lambda(\{0\})$. Last, we call μ the *splitting* measure. The latter can be *any* finite measure on $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$.

Heuristically, one can understand the coalescence as follows: each block in $\Pi(t-)$, prior to a coalescence time t, independently tosses a coin with a random parameter chosen according to $x^{-2}\Lambda(\mathrm{d}x)$. All blocks that result in heads merge, while the others remain unchanged. Furthermore, each pair of blocks coalesces at a rate $c_{\text{k}} = \Lambda(\{0\})$. This explains the binomial form in $\lambda_{n,k}$ (it can be compared with the Poisson form of $\mu_t^\lambda(k)$ for the consecutive coalescent, see Chapter 2). Regarding the fragmentation part, a block ℓ is picked “uniformly" at random prior in $[\#\Pi(t-)]$ prior to a fragmentation time t. By the assumptions of exchangeability and the absence of singletons, $\Pi_\ell(t-)$ is infinite. The new blocks within $\Pi(t)$ are formed by intersecting $\Pi_\ell(t-)$ with the blocks of an independent exchangeable partition π_f, see the Frag operator in Definition 1.A. Consequently $\#\pi_f$ blocks replace $\Pi_\ell(t-)$, inducing a jump in the block-counting process of size $\#\pi_f - 1$.

Lemma 5.2 (Proposition 2.11 in [F12]). *The block-counting process* $(N_t, t \geq 0) := (\#\Pi(t), t \geq 0)$ *of a simple EFC is a right-continuous process valued in* $\overline{\mathbb{N}} =: \mathbb{N} \cup \{\infty\}$ *with the Markov property when restricted to* $\mathbb{N}$. *Moreover the process* $\#\Pi$ *jumps from* n *to* $n-k+1$ *at rate* $\binom{n}{k}\lambda_{n,k}$, *for any* $2 \leq k \leq n$, *and from* n *to* $n+k$ *at rate* $n\mu(k)$, *for any* $k \in \mathbb{N} \cup \{\infty\}$.

We recognize here the positive jumps of an immortal branching process, see Annex A.1.2.

5.2 Coming down from infinity

Definition 5.3. Assume $\#\Pi(0) = \infty$ a.s. We say that

— the process stays infinite if

$$\forall t \geq 0 : \ \#\Pi(t) = \infty \text{ almost-surely,}$$

— the process comes down from infinity if

$$\exists t > 0 : \ \#\Pi(t) < \infty \text{ almost surely.}$$

Under the assumption $\Lambda(\{1\}) = 0$ (always in force in this chapter), as for pure coalescent processes, see Pitman [123] and [11], there is a zero-one law: either the process comes down from infinity *instantaneously* a.s., that is to say $T := \inf\{t > 0 : \#\Pi(t) < \infty\} = 0$ or it stays infinite.

Plainly if the pure Λ-coalescent process stays infinite, then any EFC process with coalescences driven by Λ stays infinite. We work therefore in this section, without loss of generality, under the assumption that the pure coalescent comes down from infinity.

5.2.1 Coming down from infinity of Λ-coalescents

A necessary and sufficient condition in this framework was discovered by Schweinsberg [131]. For any $n \geq 2$, set

$$\Phi(n) := \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k}(k-1). \tag{5.3}$$

This is the rate at which the number of blocks is decreasing when the pure Λ-coalescent process starts from n blocks.

The pure Λ-coalescent comes down from infinity if and only if

$$\sum_{n\geq 2} \frac{1}{\Phi(n)} < \infty \qquad \text{(Schweinsberg's condition).} \tag{5.4}$$

The function Φ as defined in (5.3) is not easy to study. We stress that some binomial calculations, see [131], yield the following other expression of $\Phi(n)$. For any $n \geq 2$,

$$\Phi(n) = c_{\mathrm{k}} \binom{n}{2} + \int_0^1 (nx - 1 + (1-x)^n)\, \nu_{\mathrm{Coag}}(\mathrm{d}x). \tag{5.5}$$

It is not difficult to verify, from this identity, that $(\Phi(n)/n, n \geq 1)$ is non-decreasing. One can also check analytically that $\Phi(n) \underset{n\to\infty}{\sim} \Psi(n)$ with Ψ the function of the Lévy-Khintchine form :

$$\Psi(u) = \frac{c_{\mathrm{k}}}{2} u^2 + \int_0^1 \left(e^{-xu} - 1 + ux\right) x^{-2} \Lambda(\mathrm{d}x). \tag{5.6}$$

For instance, assume that $x^{-2}\Lambda(\mathrm{d}x)=f(x)\mathrm{d}x$ with f such that $f(x)x^{2+\beta}\underset{x\to 0}{\longrightarrow} c>0$ with $\beta\in(0,1)$. Let $d:=c\frac{\Gamma(1-\beta)}{\beta(\beta+1)}$. Then, one gets by applying a Tauberian theorem that

$$\Phi(n)\underset{n\to\infty}{\sim} dn^{\beta+1}.$$

5.2.2 Coming down from infinity of simple EFCs, [F12]

Recall the definition of the splitting measure μ on $\overline{\mathbb{N}}$ in (5.2), and for any $k\geq 1$, let $\overline{\mu}(k)$ be its tail $\overline{\mu}(k):=\mu(\{k,k+1,\cdots,\infty\})=\mu(\infty)+\mu([\![k,\infty[\![)$.

Theorem 5.4 (Theorem 1.1 in [F12]). *Let $(\Pi(t),t\geq 0)$ be a simple EFC process started from an exchangeable partition such that $\#\Pi(0)=\infty$. Assume (5.4) and set*

$$\theta_\star:=\liminf_{n\to\infty}\sum_{k=1}^{\infty}\frac{n\overline{\mu}(k)}{\Phi(k+n)}\in[0,\infty] \text{ and } \theta^\star:=\limsup_{n\to\infty}\sum_{k=1}^{\infty}\frac{n\overline{\mu}(k)}{\Phi(k+n)}\in[0,\infty]. \qquad (5.7)$$

If $\theta^\star<1$ then the process comes down from infinity. If $\theta_\star>1$ then the process stays infinite.

The parameters $\theta^\star$ and $\theta_\star$ are rather intricate. When both parameters agree, namely $\theta^\star=\theta_\star$, we shall denote their common value simply by θ. A phase transition then occurs at θ, between the regime where the process stays infinite and the regime where it visits partitions with a finite number of blocks.

We stress on the fact that even when the process comes down from infinity, no assumption is made here on its possible explosion. The parameters $\theta^\star,\theta_\star$ do not deal with the accessibility of ∞. This latter question is addressed in the subsequent section.

Corollary 5.5 (Corollary 1.2 in [F12]).

1. *Assume $\mu=\lambda\delta_\infty$ and that the measure Λ satisfies (5.4). Recall $c_{\mathrm{k}}=\Lambda(0)$.*
   - *(1) If $c_{\mathrm{k}}>0$ then $\theta=2\lambda/c_{\mathrm{k}}$.*
   - *(2) If $c_{\mathrm{k}}=0$, then $\theta=\infty$ and the process stays infinite.*
2. *Assume $\mu(\infty)=0$ and that the measure Λ satisfies (5.4). If $c_{\mathrm{k}}>0$ then $\theta=0$*

Remark 5.6. The phase transition in Theorem 5.4, occurring at $\theta=2\lambda/c_{\mathrm{k}}$, is similar as the one observed in the "fast"-EFC process in [89, Theorem 1.1]. When there are binary coagulations, namely $c_{\mathrm{k}}>0$, then a finite fragmentation measure with no mass on the partitions with infinitely many blocks, i.e. $\mu(\infty)=0$, will never prevent the EFC process to come down from infinity.

Heuristics. We now give a heuristic proof of Theorem 5.4. Consider the function $f:n\mapsto\sum_{j=n+1}^{\infty}\frac{1}{\Phi(j)}$.

1. Let Π^C be a pure Λ-coalescent. Berestycki et al. have established in [12] that under (5.4), $\#\Pi^C(t)\underset{t\to 0}{\sim} v(t)$ with $v(t):=\inf\{u>0: f(u)>t\}$, the function v is known as the speed of coming down from infinity of the coalescent. Hence $f(n)$ is approximatively the time needed for the process to go below level $n+1$ when started from ∞ if no fragmentation occur.
2. Let Z be the number of blocks formed by a fragmentation event: Z has law $\mu(\cdot)/\mu(\overline{\mathbb{N}})$ and the *mean arrival time* of a fragmentation is $1/n\mu(\overline{\mathbb{N}})$.

3. By Fubini-Tonelli's theorem, one can write

$$\sum_{k=1}^{\infty}\frac{n\overline{\mu}(k)}{\Phi(n+k)} = n\mu(\overline{\mathbb{N}})\mathbb{E}\left[\sum_{k=n+1}^{n+Z}\frac{1}{\Phi(k)}\right] = \frac{\mathbb{E}[f(n)-f(n+Z)]}{1/n\mu(\overline{\mathbb{N}})}$$

and one sees that $\theta^{\star} < 1$ if and only if $\#\Pi$ jumps from n to $n+Z$ at smaller rate than it comes down from $n+Z$ to n for arbitrary large n.

While this is perhaps a convincing argument for why the parameters $\theta_{\star}, \theta^{\star}$ should be good ones, this is harder to turn this heuristics into a rigorous proof. The difficulty, lies of course, in the error made by the approximation when we say that $f(n)$ is the time needed for the pure coalescent process to come down below level $n+1$. This is correct for the Kingman coalescent (for which actually the critical case $\theta = 1$ is also handable), because of the skip-free property to the left, but not in the Λ-coalescent setting. Indeed, large negative jumps can occur and push faster the process below a given level. Moreover, fragmentations happen immediately in the process.

The proof is not based on the speed of coming down from infinity but on Lyapounov type arguments: that is to say on the search of function g satisfying for instance $\mathcal{L}g \leq -c$ in a neighbourhood of ∞, for some constant $c > 0$, where $\mathcal{L}$ denotes the generator of $\#\Pi$. Such a function allows one to get a uniform bound for the moment of the first passage times below some level. When $\theta^{\star} < 1$, the function f previously defined satisfies for large enough n, $\mathcal{L}f(n) \leq -C(1-\theta^{\star})$ with a constant $C > 0$ not depending on n. One can then show that for a large enough n, $\mathbb{E}_{\infty}(\zeta_n^-) \leq \frac{C}{1-\theta^{\star}} < \infty$ where ζ_n^- is the first passage time below level n.

Another difficulty hidden here is that the process might well go back to ∞. The arguments, sketched above, to go from the bound on the generator to the bound on the first passage time should thus not rely in any ways on assumptions about non-explosion. To circumvent this issue, the potentially exploding EFC process is shown to be monotone limit of certain non-exploding EFCs. This is reminiscent of the approach we used for defining the LCSBP, after explosion, in Chapter 3.

5.3 Explosion of the number of blocks

We now study the accessibility of ∞. This part is taken from [F13]. We assume that $\mu(\infty) = 0$ since otherwise the process clearly explodes by irreducibility. Here Schweinsberg's condition (5.4) may or may not be satisfied. Somehow symmetrically as the function Φ defined previously, we introduce a function ℓ for measuring the rate of splitting. Set for any $n \geq 1$,

$$\ell(n) := \sum_{k=1}^{n}\overline{\mu}(k). \tag{5.8}$$

Notice that $\ell(n)$ is the truncated moment of the splitting measure at level n, i.e. $\ell(n) = \mu(\mathbb{N})\mathbb{E}(Z \wedge n)$ where Z is a random variable with law $\mu(\cdot)/\mu(\mathbb{N})$. Recall Doney's condition for explosion $\sum_{n\geq 1}\frac{1}{n\ell(n)} < \infty$, see Annex A.1.2.

We first provide some general conditions on the coalescence measure Λ and the splitting measure μ ensuring that the boundary ∞ is either an exit or an entrance.

Theorem 5.7 (Theorems 3.1 and 3.3 in [F13])**.**

— *Assume that there exists some non-decreasing function g such that $\int^{\infty}\frac{\mathrm{d}x}{xg(x)} < \infty$ and*

$$\ell(n) \geq g(\log n)\log n \text{ for large enough } n. \qquad (\mathbb{H})$$

Set

$$\varrho := \limsup_{n\to\infty} \frac{\Phi(n)}{n\ell(n)}.$$

— *If* $\varrho < 1/2$, *then* ∞ *is accessible.*

— *If* (5.4) *holds and* $\varrho < 1/4$, *then* ∞ *is an exit boundary.*

— *Assume that*

$$\chi := \sum_{n=2}^{\infty} \frac{n}{\Phi(n)} \overline{\mu}(n) < \infty,$$

then ∞ *is inaccessible. If furthermore,* (5.4) *holds, then* ∞ *is an entrance boundary.*

The condition $\mathbb{H}$ implies Doney's condition and covers a rather broad class of splitting measures since for instance, all measures μ for which, for large enough n

$$\ell(n) \geq (\log^k n)^r \log^{k-1} n \times \cdots \times \log^2 n \log n,$$

with $k \geq 1$ and $r > 1$ (where $\log^k$ denotes the k-iterated logarithm), satisfy $\mathbb{H}$. We also stress that if μ satisfies $\mu(n) \underset{n\to\infty}{\sim} bn^{-(1+\alpha)}$, with $\alpha \in (0,1)$ then $\ell(n) \underset{n\to\infty}{\sim} \frac{b}{\alpha(1-\alpha)} n^{1-\alpha}$ and $\mathbb{H}$ is fulfilled.

The results in Theorem 5.7 are holding for general coalescence measures but are far from being necessary and sufficient conditions. We get however almost the full picture for the regularly varying coalescence and splitting measures. Examples for which the boundary is regular are exhibited.

Theorem 5.8 (Theorem 3.5 in [F13])**.** *Assume that* $\Phi(n) \underset{n\to\infty}{\sim} dn^{1+\beta}$ *with* $d > 0$ *and* $\beta \in (0,1)$ *and* $\mu(n) \underset{n\to\infty}{\sim} \frac{b}{n^{\alpha+1}}$ *with* $b > 0$ *and* $\alpha \in (0,\infty)$*. Then* $n\ell(n) \underset{n\to\infty}{\sim} \frac{b}{\alpha(1-\alpha)} n^{2-\alpha}$ *and the boundary* ∞ *of* $(\#\Pi(t), t \geq 0)$ *is classified as follows:*

— *if* $\alpha + \beta < 1$, *then* ∞ *is an exit boundary, here* $\theta = \infty, \varrho = 0$;

— *if* $\alpha + \beta > 1$, *then* ∞ *is an entrance boundary, here* $\theta = 0, \chi < \infty$;

— *if* $\alpha + \beta = 1$ *and further,*

  — *if* $\frac{b}{d} > \alpha(1-\alpha)$, *then* ∞ *is an exit boundary,*

  — *if* $\frac{\alpha \sin(\pi\alpha)}{\pi} < \frac{b}{d} < \alpha(1-\alpha)$, *then* ∞ *is a regular boundary,*

  — *if* $\frac{b}{d} < \frac{\alpha\sin(\pi\alpha)}{\pi}$, *then* ∞ *is an entrance boundary.*

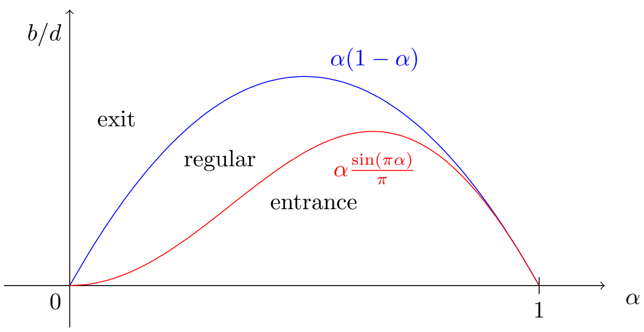


Figure 5.1 – Boundary classification when $\Phi(n) \underset{n\to\infty}{\sim} dn^{2-\alpha}$ and $\mu(n) \underset{n\to\infty}{\sim} \frac{b}{n^{1+\alpha}}$.

The study of the explosion is based on the following general sufficient condition. Let N be an irreducible minimal continuous-time Markov chain with state-space $\mathbb{N}$ and denote its generator by $\mathscr{L}$.

Theorem 5.9 (Theorem 4.1 in [F13]). *For any $a > 0$ and for any $n \in \mathbb{N}$, set $g_a(n) := n^{1-a}$ and $G_a(n) := -\frac{1}{n^{1-a}}\mathscr{L}g_a(n)$. If there exist $a > 1$ and an eventually non-decreasing positive function g satisfying $\int^{\infty} \frac{\mathrm{d}x}{xg(x)} < \infty$ such that for all large enough n*

$$G_a(n) \geq g(\log n)\log n, \tag{5.9}$$

then N explodes a.s., i.e. $\zeta_\infty := \lim_{n\to\infty} \uparrow \zeta_n^+ < \infty$ a.s., where $\zeta_n^+ := \inf\{t > 0 : N_t \geq n\}$.

Sketch of proof. The proof of Theorem 5.9 is rather technical and lengthy. Some precursor results of the same flavour can be found in Li et al. [104]. The same technique was also used in Marguet and Smadi [114]. The main idea is to use the local martingale given by

$$(N_t^{1-a}\exp\left(\int_0^t G_a(N_s)\mathrm{d}s\right), t \geq 0). \tag{5.10}$$

The latter with the assumption on G_a, (5.9), enables to control from below the probability that the process exits an interval $[\![n_0, n_1]\!]$ by n_1. In an intuitive way the integrability $\int^{\infty} \frac{\mathrm{d}x}{xg(x)} < \infty$ ensures that the event of crossing in *finite* time a sequence (n_i) going to ∞ has positive probability. □

We finally briefly explain the idea of the proof of Theorems 5.7 and 5.8. We focus on the explosion phenomenon. As in the study of the coming down from infinity, one difficulty lies in the presence of large negative jumps. We begin by investigating the process in which jumps resulting in the merging of a fixed proportion $p \in (0,1)$ of blocks are ignored. In this setting, the function G_a is decomposed into two parts (coalescence and fragmentation): $G_a = -G_a^{c,p} + G_a^f = (1 - G_a^{c,p}f/G_a^f)G_a^f$ with for any $a > 1$,

$$-G_a^{c,p}(n) \leq \frac{\Phi(n)}{n}(a-1)(1-p)^{-a} \text{ and } G_a^f(n) \geq 2^{-a}(a-1)\ell(n).$$

The condition $\mathbb{H}$, which concerns ℓ and thus G_a^f, together with the the assumption on $\varrho = \limsup_{n\to\infty}\Phi(n)/n\ell(n)$, which enables us to lower bound $1 - G_a^{c,p}f/G_a^f$, entail (5.9). This allows us to establish the accessibility of ∞ for a process whose negative jumps are bounded below. However, one can show that the latter coincides with the block-counting process on a non-empty interval of time. The method for proving the regularly varying cases is similar. In a loose sense, the crude bounds 1/2 and 1/4 are replaced by an exact explicit sum, which is easily compared to an integral and leads to the values $\frac{\alpha\sin(\pi\alpha)}{\pi}$ in Theorem 5.8. The value $\alpha(1-\alpha)$ delimiting the region where the process stays infinite or not is obtained by Theorem 5.4.

5.4 Comments

Scope and limitations of the results

— It appears that in order to deal with the critical cases, i.e. when b/d lies on the curves in Figure 5.1, one would need to work with a sharper function than g_a. The behavior of the coalescence measure $x^{-2}\Lambda(\mathrm{d}x)$ close to 1 seems also to play a role in this case and we have not pushed further the study. Similarly as what we have explained in the comments section of Chapter 3, we could compare the speeds of explosion of the pure stable branching with the speed of coming down of the coalescent. Unsurprisingly when $\beta = 1 + \alpha$ they are of the same order. However, as for the LCSBPs, it is not sufficient to consider separately the speeds in order to deduce the different regimes given in Theorem 5.8.

— The assumption on the fragmentation measure to give no mass to the singletons, see Definition 1.B is merely technical. It is used for instance in order to define some "couplings" of the EFC processes started from partitions with different numbers of blocks. For more details, see Remark 3.13 in [F12]. Other couplings can be designed in the presence of dust, see Kyprianou et al. [89]. There is no difference at the level of the number of blocks between the case with dust and that of a fragmentation measure giving mass to set of infinite proper partitions (i.e. with no singletons).

— EFCs whose splitting and coalescence measures have functions $\Phi(n)$ and $\ell(n)$ of the order $n(\log(n))^{\alpha}$ with $\alpha > 1$ have been studied in [F13]. We have not found examples for which ∞ is regular in this setting.

— Most of the arguments designed in [F12] and [F13] would also hold when the coalescences are driven by the so-called regular Ξ-coalescents (multiple mergings can then happen simultaneously), see Schweinsberg [130] and e.g. [F2] for the notion of regular Ξ and two different proofs for the necessary part of Schweinsberg's condition for Ξ's.

Ouverture

Other functionals of EFCs than their number of blocks could be considered. We may think for instance about the largest block and its mass.

Last, and this is no so much related to the EFC processes, but to the best of my knowledge no *necessary* and sufficient condition for the coming down from infinity has been established beyond the regular case. We refer to Möhle and Herriger [116] for a work in this direction.

CHAPTER 6

Boundary classification of Λ-Wright-Fisher processes with selection

Summary.

In this last chapter, based on [F16], we will study a fundamental mathematical model in population genetics called Wright-Fisher processes. They are Markov processes, taking their values in the interval $[0, 1]$ and representing the frequency over time of an allele (or type) in a population of fixed size which evolves by resampling. We consider pure-jump Wright-Fisher (WF) processes in continuous time and space and generalise them by taking into account an extra force of selection. Selection dynamics are typically modelled deterministically, so that the frequency of a type evolves both due to the resampling and due to a frequency-dependent term modeling how deleterious the allele considered is.

When there is no selection, those processes are known to satisfy a moment duality with the Λ-coalescents. This duality allows one for instance to establish that a necessary and sufficient condition for the process to be absorbed at 1 in finite time with positive probability is that the coalescent process comes down from infinity. When the selection takes a particular form, such a moment duality will be satisfied at the *level of the generators* between the WF process with selection and the block-counting process of a simple EFC. Since the latter has different behaviors at its boundary ∞, the same will be true for the Λ-Wright-Fisher process with selection at its boundary 1.

In a reminiscent way as what we did in Chapter 3, we will be able to construct an extension of the process "beyond fixation" of the disadvantaged allele using the duality relationship and the results of Chapter 5. There will be actually two different moment dualities. One for the *minimal* WF process (which is forced to get absorbed at its boundaries if they are accessible) and one for the extended process. The dual processes will be respectively the *unstopped* block-counting process of an EFC and the *stopped* one.

6.1 A stochastic equation

Let Λ be a finite measure over $[0,1]$. Let μ be a finite measure on $\mathbb{N} := \{1,2,\ldots\}$. Denote by f the generating function of the probability measure $\xi(\cdot) = \mu(\cdot)/\mu(\mathbb{N})$ over $\mathbb{N}$, for all $x \in [0,1]$, $f(x) := \sum_{k=1}^{\infty} x^k \xi(k)$ and set $\sigma = \mu(\mathbb{N})$. Consider the stochastic equation

$$F_t(x) = x + \int_0^t \int_0^1 \int_0^1 z \left(\mathbb{1}_{\{v \le F_{s-}(x)\}} - F_{s-}(x)\right)\overline{\mathcal{M}}(\mathrm{d}s, \mathrm{d}v, \mathrm{d}z) \\ - \sigma \int_0^t F_s(x)(1 - f(F_s(x)))\mathrm{d}s, \tag{6.1}$$

where $\mathcal{M}$ is a Poisson point process on $\mathbb{R}_+ \times [0,1] \times [0,1]$ with intensity $m(\mathrm{d}t, \mathrm{d}v, \mathrm{d}z) = \mathrm{d}t\mathrm{d}vz^{-2}\Lambda(\mathrm{d}z)$ and $\overline{\mathcal{M}}$ stands for the compensated measure $\overline{\mathcal{M}} = \mathcal{M} - m$. Notice that both terms vanishes when the process reaches 0 or 1 since $f(0) = 0$, $f(1) = 1$. Moreover for all $x \in [0,1]$, $1 - f(x) \ge 0$ so that the drift term in (6.1) is negative. In the general case, Λ-Wright-Fisher processes may have a diffusion part. We focus here on the case of a measure Λ on $[0,1]$ with no mass at 0 or at 1.

Any process $(F_t(x), t \ge 0)$ solution to the equation (6.1) is valued in $[0,1]$. Imagine a population of constant size 1, whose individuals carry at any time one allele among a set of two alleles $\{a, A\}$. Suppose that the process $(F_t(x), t \ge 0)$ follows the frequency of allele a when initially the proportion of individuals carrying allele a is of size x. Before reaching boundaries, the time-dynamics of $(F_t(x), t \ge 0)$ consists of two parts:

— the resampling which is governed by the Poisson random measure $\mathcal{M}$: for any (t, v, z) atom of $\mathcal{M}$,

  — if $v \le F_{t-}(x)$, then allele a is sampled and a fraction $z \in (0,1)$ of the alleles A at time $t-$ is replaced by the allele a at time t. The frequency of allele a increases:
  $$F_t(x) = z(1 - F_{t-}(x)) + F_{t-}(x),$$

  — if $v > F_{t-}(x)$, then allele A is sampled and a fraction $z \in (0,1)$ of the alleles a at time $t-$ is replaced by the allele A at time t. The frequency of allele a decreases:
  $$F_t(x) = (1 - z)F_{t-}(x),$$

— the selection which is modeled by function f characterizing the disadvantage of allele a: the frequency of allele a decreases continuously in time along the negative deterministic drift:
$$-\sigma F_t(x)(1 - f(F_t(x)))\mathrm{d}t.$$

When $\sigma = 0$, the drift term in (6.1) governing the selection disappears and the solution of (6.1) becomes the classical Λ-Wright-Fisher process, see Bertoin and Le Gall [22] and Dawson and Li [43], which represents the evolution of the frequency of a *neutral* allele (or type) in a two-allele model evolving by resampling. In particular, when there is no selection term, the SDE (6.1) has a pathwise unique strong solution and the boundaries 0 and 1 are both absorbing whenever they are reached. The event of absorption at 1 (respectively at 0) is called *fixation* of the allele a (respectively A) in the genetics terminology. It corresponds to the fact that all individuals have a common allele from a finite time almost surely. We refer the reader to, for instance, Etheridge's Saint Flour lecture notes [52].

Call $(N_t^{(n)}, t \geq 0, n \in \mathbb{N})$ the block counting process of a Λ-coalescent started from n blocks (see Chapter 5). One has for all $x \in [0,1]$ and $n \in \mathbb{N}$,

$$\mathbb{E}[F_t(x)^n] = \mathbb{E}[x^{N_t^{(n)}}]. \tag{6.2}$$

By letting n go to ∞ in the identity (6.2), we see that fixation at 1 occurs if and only if the Λ-coalescent comes down from infinity i.e.

$$\int^{\infty} \frac{\mathrm{d}x}{\Phi(x)} < \infty, \tag{6.3}$$

with $\Phi(x) := \Phi([x])$ for any $x \geq 2$, see Section 5.2.1.

One of the first models generalizing the Λ-Wright-Fisher process by incorporating selection is perhaps the case for which $f(x) = x$ and the drift term in the SDE (6.1) takes the form $-\sigma F_s(x)(1 - F_s(x))\mathrm{d}s$. In this setting the measure μ reduces to a Dirac mass at 1 with weight σ. Such processes have been studied by numerous authors, we refer for instance to Baake et al. [7], Bah and Pardoux [8], Etheridge and Griffiths [54], Griffiths [70] and Foucart [F4]. Bah and Pardoux [8, Theorem 4.3] have established that in this case, fixation at 0 or 1 occurs almost surely if and only if (6.3) is satisfied. In particular, when (6.3) holds, despite that allele a is deleterious when $\sigma > 0$, the population still has a positive probability to get fixed on allele a in a finite time almost surely.

Returning to the Equation (6.1), the behavior of the positive function $x \mapsto 1-f(x)$ near 1 actually reflects the strength of the selective advantage of allele A over a. One question we are addressing is to see whether a selection term can overcome the Λ-resampling mechanism and prevent fixation of the deleterious allele a.

When f is Lipschitz on $[0,1]$, i.e. $f'(1-) < \infty$, fundamental results on SDEs with jumps, see e.g. [43], entail that there exists a unique strong solution to (6.1). Moreover pathwise uniqueness holds and since 1 is always a solution, it entails that the process is absorbed at 1 if it reaches it. Actually in this case, fixation at boundary 1 is always possible when (6.3) holds.

When the drift term in (6.1) is non-Lipschitz at 1, namely $f'(1-) = \infty$, pathwise uniqueness of the solution to the SDE (6.1) might not hold. In this case the only solution to (6.1) whose existence and uniqueness is guaranteed is the minimal one, $(X_t^{\min}, t \geq 0)$, which is stopped upon reaching boundary 1. Several weak solutions to (6.1) with different behaviors at boundary 1 may exist. In the pure deterministic fashion, notice that the function F solution to (6.1) (with only the drift part) would be able to start from 1 if and only if $\int^{1-} \frac{\mathrm{d}x}{1-f(x)} < \infty$ [1].

Let $(F_t^{\mathrm{r}}(x), t \geq 0, x \in [0,1])$ be a process valued in $[0,1]$. For any $x \in [0,1]$, let τ_1 be the first hitting time of boundary 1, i.e. $\tau_1 := \inf\{t > 0 : F_t^{\mathrm{r}}(x) = 1\} \in [0,\infty]$. The process $(F_t^{\mathrm{r}}, t \geq 0)$ is said to be an extension of the minimal process $(F_t^{\min}, t \geq 0)$, if $(F_{t\wedge\tau_1}^{\mathrm{r}}, t \geq 0)$ has the same law as $(F_t^{\min}, t \geq 0)$.

6.2 Moment dualities and boundaries correspondences

We denote by $(N_t^{(n)}, t \geq 0)$ the block-counting process of a simple EFC started from n blocks with coalescence measure Λ and splitting measure μ. We refer to Chapter 5 for more details.

1. which should be compared to Dynkin's condition for explosion of immortal branching process, see Annex A.1.2

We first state a duality relationship which holds for any minimal Λ-Wright-Fisher process with frequency-dependent selection, subject to the condition $\Lambda(\{1\}) = 0$. No assumption on the generating function f is made [2].

Theorem 6.1 (Theorem 2.1 in [F16]). *The Markov process $(F_t^{\min}(x), t \geq 0, x \in [0,1])$ satisfies the following property. For any $t \geq 0$, $x \in [0,1]$ and any $n \in \mathbb{N}$*

$$\mathbb{E}[F_t^{\min}(x)^n] = \mathbb{E}[x^{N_t^{(n)}}]. \tag{6.4}$$

In particular,

$$\mathbb{P}(F_t^{\min}(x) = 1) = \lim_{n\to\infty} \mathbb{E}[x^{N_t^{(n)}}] = \mathbb{E}[x^{N_t^{(\infty)}}] \in [0,1]$$

and the process $(F_t^{\min}(x), t \geq 0, x \in [0,1])$ gets absorbed at 1 with positive probability if and only if the process $(N_t^{(\infty)}, t \geq 0)$ comes down from infinity (i.e. ∞ is either an entrance or a regular non-absorbing boundary).

We deduce from Theorem 6.1 two important results for the block counting process of a simple EFC process. Those results were left unaddressed in the previous chapter.

Theorem 6.2 (Markov property of $(\#\Pi(t), t \geq 0)$, Theorem 2.3 in [F16]). *Let $(\Pi(t), t \geq 0)$ be a simple EFC process whose coalescence measure is Λ and splitting measure is μ. The block counting process $(N_t, t \geq 0) := (\#\Pi(t), t \geq 0)$ with state-space $\overline{\mathbb{N}}$ is a Markov process satisfying the Feller property.*

Theorem 6.3 (Recurrence of $(\#\Pi(t), t \geq 0)$, Theorem 2.4 in [F16]). *If the process $(\#\Pi(t), t \geq 0)$ comes down from infinity (e.g. if $\theta^\star < 1$), then it is positive recurrent and has a stationary distribution whose generating function is $\varphi : x \in [0,1] \mapsto \mathbb{P}_x(\tau_1 < \tau_0)$.*

We now introduce the extended process $(F_t^{\mathrm{r}}, t \geq 0)$. The latter is constructed in the following way. We first look at processes, solution to the Equation (6.1) with a defective generating function f such that $1 - f(1) = \lambda > 0$. This causes an additional drift term $-\lambda F_t \mathrm{d}t$, which can be seen as modeling *mutation* from the deleterious allele a to the advantaged one A.

Under the assumption that there is no Kingman component, i.e. $\Lambda(\{0\}) = 0$, those processes, called F^λ's, can all be started from boundary 1. This is simply due to the fact that the process $(N_t, t \geq 0)$ has its boundary ∞ exit when there is no Kingman part, see Corollary 5.5 in the previous Chapter.

Our core object of study is the limit process that arises when the parameter λ tends to 0 (i.e. the mutation rate becomes very low). Hence define formally the limit process F^{r} as

$$F_t^{\mathrm{r}} := \lim_{\lambda\to 0+} F_t^\lambda \text{ for all } t \geq 0. \tag{6.5}$$

Theorem 6.4 (Theorem 2.7 in [F16]). *The convergence (6.5) holds in Skorokhod topology (and a.s. for fixed times). Moreover the Feller process $(F_t^{\mathrm{r}}(x), t \geq 0, x \in [0,1])$ is extending the minimal process such that for any $t \geq 0$ and any $n \in \mathbb{N}$,*

$$\mathbb{E}[F_t^{\mathrm{r}}(x)^n] = \mathbb{E}[x^{N_t^{\min,(n)}}] \text{ for any } x \in [0,1) \text{ and } \mathbb{E}[F_t^{\mathrm{r}}(1)^n] = \mathbb{P}(N_t^{\min,(n)} < \infty), \tag{6.6}$$

where $(N_t^{\min,(n)}, t \geq 0) := (N_{t\wedge\zeta_\infty}^{(n)}, t \geq 0)$ for $\zeta_\infty := \inf\{t > 0 : N_t^{(n)} = \infty\} \in [0,\infty]$.

2. in particular it can be defective, i.e. ξ in (6.1) can be a subprobability

Furthermore, by combining the duality relationships displayed in Theorem 6.1 and Theorem 6.4:

$$F^{\min} \overset{\text{moment dual}}{\longleftrightarrow} N \text{ and } F^{\mathrm{r}} \overset{\text{moment dual}}{\longleftrightarrow} N^{\min},$$

we see that the boundary 1 for $F^{\min}$ (and hence for F^{r}) is accessible if and only if ∞ is non-absorbing for N, and 1 is non-absorbing for F^{r} if and only if ∞ is accessible for $N^{\min}$ (and hence for N). To sum up, the two dualities yield Table 6.1.

Boundary 1 of F^{r}	Boundary ∞ of N
entrance	exit
regular	regular
exit	entrance
natural	natural

Table 6.1 – Classification of boundaries.

We stress that in the regular case, see the second line of Table 6.1 above, if one process has its boundary regular *non-absorbing* then its dual process has necessarily its boundary regular *absorbing*. This is reminiscent to the classification we found for the one-dimensional diffusions, see Theorem 3.16, as weel as for the logistic CSBP, with boundary ∞ regular, see in particular Theorem 3.11.

Since there are several possible ways to leave a regular boundary, Table 6.1 does not specify completely the behavior of the process at the boundary when it is regular non-absorbing. Recall that a regular boundary is said to be *reflecting* when the set of times at which the process lies at the boundary, has a zero Lebesgue measure. A regular boundary is also said to be *regular for itself* if the process started from the boundary returns immediately to it almost surely. In the same fashion, we classify the exit and entrance boundaries by saying that boundary 1 is an *instantaneous entrance* if it is an entrance and the first entrance time in $[0,1)$, $\tau^1 := \inf\{t > 0 : F^{\mathrm{r}}_t(x) < 1\}$ satisfies $\mathbb{P}_1(\tau^1 = 0) = 1$. The boundary ∞ is an *instantaneous exit* if it is an exit and the first explosion time $\zeta_\infty := \inf\{t > 0 : N^{(n)}_{t-} \text{ or } N^{(n)}_t = \infty\}$ satisfies for any $t > 0$, $\mathbb{P}_n(\zeta_\infty \leq t) \longrightarrow 1$, as n goes to ∞. Similar definitions hold for instantaneous exit boundary 1 and instantaneous entrance boundary ∞.

The next theorem explains the possible behaviors of the dual processes at their boundaries when they are regular non-absorbing.

Theorem 6.5 (regular reflecting/regular for itself, Theorem 2.8 in [F16])**.** *One has the following table:*

Boundary 1 of F^{r}	Boundary ∞ of N
regular reflecting	regular for itself
regular for itself	regular reflecting
instantaneous entrance	instantaneous exit
instantaneous exit	instantaneous entrance

Table 6.2 – regular for itself/regular reflecting

We address now the long-term behavior of the extended Λ-Wright-Fisher process with selection $(F_t^{\mathrm{r}}, t \geq 0)$. Recall that by fixation, we mean that all individuals get one of the two alleles and keep it forever. When 1 is an exit, fixation of the deleterious allele has a positive probability to occur. When the boundary 1 is regular non-absorbing or entrance, fixation at 1 can not occur, and we shall actually see that there is almost sure fixation of the advantageous allele.

Theorem 6.6 (Theorem 2.10 in [F16]). *Assume that Λ satisfies (6.3). If $(F_t^{\mathrm{r}}, t \geq 0)$ has boundary 1 either regular non-absorbing or an entrance, then for all $x \in [0,1]$,*

$$\exists\ t_0 > 0; F_t^{\mathrm{r}}(x) = 0 \text{ for all } t \geq t_0,\ \text{a.s.}$$

Table 6.1 and Table 6.2 provide a theoretical classification of the boundaries. We now give explicit sufficient conditions on the resampling measure Λ and the selection function f for each possible boundary behavior. They are obtained via the correspondences stated in Table 6.1 and Table 6.2, by transferring previous results on the boundary ∞ of the block counting process N, obtained in Chapter 5 to results on the boundary 1 of F^{r}.

The next theorem provides conditions over the selection function f and the resampling measure Λ for 1 to be an absorbing boundary for the (non-stopped) process $(F_t^{\mathrm{r}}, t \geq 0)$, so that 1 is either an exit or a natural boundary.

Theorem 6.7 (Theorem 2.11 in [F16]). *If f is Lipschitz on $[0,1]$, or if $x \mapsto 1 - f(x)$ is regularly varying at 1 with index $\alpha \in [0,1)$ and satisfies*

$$\int^{1-} \frac{1 - f(x)}{(1-x)^3 \Phi(1/(1-x))} \mathrm{d}x < \infty, \tag{6.7}$$

then the boundary 1 of $(F_t^{\mathrm{r}}, t \geq 0)$ is absorbing. Assume (6.7) holds true, then

i) if $\int^{\infty} \frac{\mathrm{d}z}{\Phi(z)} < \infty$, 1 is an instantaneous exit boundary;

ii) if $\int^{\infty} \frac{\mathrm{d}z}{\Phi(z)} = \infty$, 1 is a natural boundary.

We now provide a sufficient condition on the resampling measure Λ and the selection function f entailing that the process $(F_t^{\mathrm{r}}, t \geq 0)$, has boundary 1 as an entrance. Introduce the following condition over the function $x \mapsto 1 - f(x)$:

Condition $\mathcal{H}$: there exists a positive function L defined on $(0,1)$ such that $\int^{1-} \frac{1}{L(x)} \mathrm{d}x < \infty$, the map $h : x \mapsto \frac{L(x)}{(1-x)\log(1/(1-x))}$ is eventually non-decreasing in the neighbourhood of 1 and

$$1 - f(x) \geq L(x) \text{ for } x \text{ close enough to } 1.$$

Remark 6.8. Condition $\mathcal{H}$ encompasses a regularity assumption on the difference quotient of the function f near 1. Indeed the condition on the map h holds if the function $x \mapsto (1 - f(x))/(1-x)\log(1/(1-x))$ stays above a non-decreasing function in some neighbourhood of 1. In this case, Condition $\mathcal{H}$ reduces to $\int^{1-} \frac{\mathrm{d}x}{1-f(x)} < \infty$. Of course, $\mathcal{H}$ is intimately connected to the condition $\mathbb{H}$ in Chapter 5.

Theorem 6.9 (Theorem 2.12 in [F16]). *Assume Condition $\mathcal{H}$ holds. If*

$$\frac{(1-x)^2 \Phi(1/\log(1/x))}{1 - f(x)} \underset{x \to 1-}{\longrightarrow} 0, \tag{6.8}$$

then the boundary 1 is an instantaneous entrance boundary.

Theorem 6.10 (Theorem 2.16 in [F16]). *Let $\alpha, \beta \in (0,1)$ and $\sigma, \varrho > 0$. Assume*

$$\Lambda(\mathrm{d}x) = h(x)\mathrm{d}x \text{ with } h(x) \underset{x\to 0+}{\sim} \varrho x^{-\beta} \text{ and } \sigma(1-f(x)) \underset{x\to 1-}{\sim} \sigma(1-x)^{\alpha}. \tag{6.9}$$

The boundary 1 of $(F_t^{\mathrm{r}}, t \geq 0)$ is classified as follows :

i) if $\alpha + \beta < 1$, then 1 is an instantaneous entrance;

ii) if $\alpha + \beta > 1$, then 1 is an instantaneous exit;

iii) if $\alpha + \beta = 1$ and further,

— *if $\sigma/\varrho > \frac{\pi}{(2-\alpha)\sin(\pi\alpha)}$, then 1 is an instantaneous entrance;*

— *if $\frac{1}{(1-\alpha)(2-\alpha)} < \sigma/\varrho < \frac{\pi}{(2-\alpha)\sin(\pi\alpha)}$, then 1 is regular reflecting;*

— *if $\sigma/\varrho < \frac{1}{(1-\alpha)(2-\alpha)}$, then 1 is an instantaneous exit.*

Remark 6.11. Cases i) and ii) are consequences of Theorem 6.9 and Theorem 6.7. Important examples for which the condition (6.9) hold are coalescence measures Λ of the Beta form, $\Lambda(\mathrm{d}x) = \varrho x^{-\beta}(1-x)^{a-1}\mathrm{d}x$ for $\beta \in (0,1)$ and $a > 0$, and generating functions f associated to Sibuya distribution, $f(x) = 1-(1-x)^{\alpha}$ for $\alpha \in (0,1)$.

Theorem 6.12 (Theorem 2.18 in [F16]). *Let $\sigma, \varrho > 0$ and $\alpha \in (0,1)$. Assume*

$$\Lambda(\mathrm{d}x) = h(x)\mathrm{d}x \text{ with } h(x) \underset{x\to 0+}{\sim} \varrho x^{-(1-\alpha)} \text{ and } \sigma(1-f(x)) \underset{x\to 1-}{\sim} \sigma(1-x)^{\alpha}.$$

If $\frac{1}{(1-\alpha)(2-\alpha)} < \sigma/\varrho < \frac{\pi}{(2-\alpha)\sin(\pi\alpha)}$, then the extended process $(F_t^{\mathrm{r}}, t \geq 0)$ has its boundary 1 regular for itself.

Remark 6.13. Since the process $(F_t^{\mathrm{r}}, t \geq 0)$ is Feller, when boundary 1 is regular reflecting and regular for itself, standard theory, see e.g. [16, Chapter IV] ensures the existence of a local time of the process $(F_t^{\mathrm{r}}, t \geq 0)$ at 1 whose inverse subordinator has no drift.

By combining Theorem 6.12 and Theorem 6.5, we obtain the following corollary for the block counting process $(N_t, t \geq 0)$ of a simple EFC process $(\Pi(t), t \geq 0)$ whose splitting measure μ and coalescence measure Λ are regularly varying. Recall that by Theorem 6.2, $(N_t, t \geq 0) := (\#\Pi(t), t \geq 0)$ is a Markov process with state-space $\bar{\mathbb{N}}$. The first assertion i) below specifies the behavior of $(N_t, t \geq 0)$ when its boundary ∞ is regular.

Corollary 6.14 (Corollary 2.20 in [F16]). *Let $\alpha \in (0,1)$. Assume $\Phi(n) \underset{n\to\infty}{\sim} dn^{2-\alpha}$ with $d > 0$ and $\mu(n) \underset{n\to\infty}{\sim} \frac{b}{n^{1+\alpha}}$ with $b > 0$.*

i) If $\frac{\alpha\sin(\pi\alpha)}{\pi} < b/d < \alpha(1-\alpha)$, then the boundary ∞ of the process $(\#\Pi(t), t \geq 0)$ is regular reflecting.

ii) If $b/d < \alpha(1-\alpha)$, then the process $(\#\Pi(t), t \geq 0)$ is positive recurrent and admits a stationary distribution carried over $\mathbb{N}$.

6.3 Comments

— The duality used here relates only the functional of the number of blocks of a simple EFC with the Λ-Wright-Fisher process with selection. One may wonder if there is a duality at the level of the partitions and measures, as it is the case *without* selection between the generalized Fleming-Viot processes and the partition-valued simple exchangeable coalescents, see e.g. [18].

— As mentioned at the beginning of the chapter, the process in (6.1) is a generalization of the case whose drift term is of *logistic* form $-\sigma F_s(x)\,(1 - F_s(x))$. In a similar fashion, but in the setting of CSBPs instead of Wright-Fisher processes, one could generalize the quadratic drift term of the logistic CSBP by a drift of the form "$-\Phi(Z_t)\mathrm{d}t$" with Φ a Lévy-Khintchine function. This topic is currently studied by a PhD student.

— A natural other question would be to see whether an analogue structure as the collisions, as defined in Chapter 4, can be designed in the framework of Λ-Wright-Fisher processes. We refer the reader to Gonzalez et al. [67] for some results in this direction.

Annexes

A Basics on branching processes

We first recall well-known material on continuous-time Bienaymé-Galton-Watson processes. We refer to Athreya and Ney [6, Chapter III] for more details on these processes.

A.1 Continuous-time branching Markov chains

Those processes (possibly in their inhomogeneous-time version) appear in our study of the genealogy of the continuous-state branching processes (CSBPs), see Chapter 2. We also encounter them while studying the number of blocks in the exchangeable fragmentation-coalescence processes and the Wright-Fisher processes, see Chapter 5.

A.1.1 Definition

Consider a finite measure ν on $\overline{\mathbb{Z}}_+ := \mathbb{Z}_+ \cup \{\infty\}$ such that $\nu(1) = 0$. A continuous-time Galton-Watson process starting from n, of reproduction measure ν, $(Z_t(n), t \geq 0)$, is the process counting the number of individuals in a random population with n ancestors where all individuals behave independently, each of them has an exponential lifespan ζ with parameter $\nu(\overline{\mathbb{Z}}_+)$ and generates at its death a random number of children (possibly infinite) with probability distribution $\nu/\nu(\overline{\mathbb{Z}}_+)$.

This continuous-time Markov chain has for generator

$$\mathscr{L}^b f(n) = n \sum_{k \in \overline{\mathbb{Z}}_+} (f(n+k-1) - f(n))\nu(k)$$

with $n \in \overline{\mathbb{Z}}_+$ and where f is any function with a limit at ∞. Let $x \in [0,1]$ and $f_x(n) = f_n(x) = x^n$, one has

$$\mathscr{L}^b f_x(n) = -nx^{n-1}\psi(x) = -\psi(x)\frac{\mathrm{d}}{\mathrm{d}x} f_n(x)$$

with ψ the function

$$\psi(x) := -\sum_{k \in \overline{\mathbb{Z}}_+} (x^k - x)\nu(k), \qquad x \in [0,1] \tag{A.10}$$

and the convention $x^\infty = 0$ if $x < 1$ and $1^\infty = 1$. The process Z is characterized in law by ν and thus by ψ. By its definition, the process $(Z_t(n), t \geq 0)$ satisfies the branching property

$$\forall n, m \geq 0, \forall t \geq 0,\ Z_t(n+m) \overset{(d)}{=} Z_t^{(1)}(n) + Z_t^{(2)}(m), \tag{A.11}$$

where $Z_t^{(1)}(n)$ and $Z_t^{(2)}(m)$ are independent copies of $Z_t(n)$ and $Z_t(m)$. This entails that the generating function of $Z_t(n)$ for any $t \geq 0$ has the form

$$\mathbb{E}[x^{Z_t(n)}] = f_t(x)^n, \qquad x \in [0,1], n \in \mathbb{Z}_+ \tag{A.12}$$

where for all $t \geq 0$ and $s \in (0,1)$, $t \mapsto f_t(s)$ is the solution of $\int_{f_t(s)}^s \frac{\mathrm{d}z}{\psi(z)} = t$ for any $t \geq 0$.

A.1.2 Boundaries of branching processes

The branching property entails that the boundaries 0 and ∞ are always absorbing, i.e. the process stays at the boundary in case the latter is reached. When $\nu(0) > 0$, the process has a positive probability to reach 0 and stays there after hitting it. If the process starts from n individuals, this probability is given by p^n where p is the largest zero of ψ.

When $\sum_{k\in\overline{\mathbb{Z}}+} k\nu(k) \leq \nu(\overline{\mathbb{Z}}+)$, the process is said to be critical or subcritical (in the case of strict inequality) and extinction is almost certain, i.e. $p = 1$. In the supercritical case, that is to say $\sum_{k\in\overline{\mathbb{Z}}+} k\nu(k) > \nu(\overline{\mathbb{Z}}+)$, either $p > 0$ or $p = 0$. The latter case occurs if and only if ν has no mass at 0. In this case, we say that the process is *immortal*. More precisely, each individual has at least two children and $(Z_t(n), t \geq 0)$ is non-decreasing in time. In this context, we sometimes prefer to work with the measure $\mu(k) = \nu(k+1)$, in which case the generator takes the form

$$\mathscr{L}^b f(n) = n \sum_{k\in\overline{\mathbb{N}}} (f(n+k) - f(n))\mu(k). \tag{A.13}$$

We call μ the splitting measure. Notice that ψ can be rewritten as

$$\psi(x) = -x \sum_{j\in\overline{\mathbb{N}}} (x^j - 1)\mu(j) = -\mu(\overline{\mathbb{N}})x(1 - f(x))$$

with f the generating function of $\xi(\cdot) = \mu(\cdot)/\mu(\overline{\mathbb{N}})$.

When $\mu(\infty) > 0$, the process goes to ∞ by a jump and stays there. Consider an *immortal* pure branching process with splitting measure μ. Any immortal process increases almost surely towards ∞. It is well-known that some of branching processes can explode in finite time (i.e. reach ∞) even if $\mu(\infty) = 0$. Denote by ζ_∞ the explosion time: $\zeta_\infty := \inf\{t > 0 : Z_t = \infty\}$. Set $\ell(n) := \sum_{k=1}^{n} \overline{\mu}(k)$ for any $n \geq 1$. One has the two following equivalent conditions for explosion, see Harris' book [3] [73, Chapter V, Section 9, Theorem 9.1] and Doney [45]

$$\begin{aligned} \zeta_\infty < \infty \text{ a.s.} &\Longleftrightarrow \int^1 \frac{\mathrm{d}x}{|\psi(x)|} < \infty \quad \text{(Dynkin's condition)} \\ &\Longleftrightarrow \sum_{n\geq 1} \frac{1}{n\ell(n)} < \infty \quad \text{(Doney's condition)}. \end{aligned}$$

Notice that Doney's condition only involves the splitting measure. Plenty examples of exploding processes can be constructed from Doney's condition. This was used in particular in Chapter 5 and Chapter 6.

A.2 Continuous-state branching processes

The continuous-state branching processes (CSBPs) have been first introduced by Jirina [136] and Lamperti [97]. They arise as the scaling limits of discrete Bienaymé-Galton-Watson chains, see Grimwall [71], Lamperti [98]. We refer also the reader to Le Gall's lecture note [101, Chapter II] where they are constructed (following the arguments of the proof of [136, Theorem 4]) as limits of continuous-time branching Markov chains.

They are also characterized as the class of Markov processes, with state-space $[0, \infty]$, the one-point compactification of $[0, \infty)$, that are satisfying the branching property

$$\forall z, z' \in [0, \infty], \; \forall t \geq 0, \; X_t(z + z') \overset{d}{=} X_t^{(1)}(z) + X_t^{(2)}(z') \tag{A.14}$$

where $X_t^1(z)$ and $X_t^2(z')$ are independent copy of $X_t(z)$ and $X_t(z')$. We refer to Silverstein [136] and Li's book [106].

3. Harris cites Dynkin for this criterion

Moreover, for any CSBP $(X_t, t \geq 0)$, there exists a function $(u_t(x), t \geq 0)$, called the *cumulant*, and some function Ψ, called *the branching mechanism*, such that for all $t \geq 0$, $z \geq 0$ and $x \in (0, \infty)$,

$$\mathbb{E}_z[e^{-xX_t}] = e^{-zu_t(x)} \text{ with } \frac{\mathrm{d}}{\mathrm{d}t}u_t(x) = -\Psi(u_t(x)), u_0(x) = x. \tag{A.15}$$

The function Ψ is of the Lévy-Khintchine form:

$$\Psi(z) = -\lambda + \frac{\sigma^2}{2}z^2 + \gamma z + \int_0^{+\infty} \left(e^{-zx} - 1 + zx\mathbb{1}_{\{x\leq 1\}}\right)\pi(\mathrm{d}x), \tag{A.16}$$

with $\lambda \geq 0, \gamma \in \mathbb{R}$, $\sigma \geq 0$, and π a Lévy measure on $(0, \infty)$.

A.2.1 A stochastic equation

A CSBP with branching mechanism Ψ is by now often introduced as the unique strong solution to a stochastic equation, see e.g. Li's lecture notes [108]. We state it here in the framework of a flow: define for each $s \geq 0$, the process $(X_{s,t}(z), t \geq s, z \geq 0)$ solution to

$$\begin{aligned} X_{s,t}(z) = \; & z + \sigma \int_s^t \int_0^{X_{s,r-}(z)} \widetilde{B}(\mathrm{d}r, \mathrm{d}u) + \gamma \int_s^t X_r \mathrm{d}r + \int_s^t \int_0^{X_{s,r-}} \int_{(0,1]} h\overline{\mathcal{N}}(\mathrm{d}r, \mathrm{d}u, \mathrm{d}h) \\ & + \int_s^t \int_0^{X_{s,r-}} \int_{(1,\infty]} h\mathcal{N}(\mathrm{d}r, \mathrm{d}u, \mathrm{d}h). \end{aligned} \tag{A.17}$$

Moreover the following flow property holds: $\forall 0 \leq s \leq r$, almost surely

$$(X_{s,t}(z), t \geq r) = (X_{r,t}(X_{s,r}(z)), t \geq r).$$

The ingredients of (A.17) are specified as follows: $\widetilde{B}$ is a white noise on $(0, \infty) \times (0, \infty)$ based on the Lebesgue measures $\mathrm{d}r\mathrm{d}u$, $\mathcal{N}(\mathrm{d}r, \mathrm{d}u, \mathrm{d}h)$ is an independent Poisson random measure on $[0, \infty)^2 \times (0, \infty]$ with intensity $\mathrm{d}r\mathrm{d}u\pi(\mathrm{d}h)$ and by convention $\lambda = \pi(\infty) \geq 0$. The signed random measure $\overline{\mathcal{N}}$ stands for the compensated random measure,

$$\overline{\mathcal{N}}(\mathrm{d}r, \mathrm{d}u, \mathrm{d}h) := \mathcal{N}(\mathrm{d}r, \mathrm{d}u, \mathrm{d}h) - \mathrm{d}r\mathrm{d}u\pi(\mathrm{d}h).$$

The first stochastic integral with respect to $\widetilde{B}$ is a rewriting of the Feller diffusive part. Namely for a fixed initial value z,

$$\int_s^t \int_{[0, X_{s,r-}]} \widetilde{B}(\mathrm{d}r, \mathrm{d}u) = \int_s^t \sqrt{X_r}\mathrm{d}B_r,$$

for some Brownian motion B. We refer to Li and Ma [110, page 940] and Dawson and Li [43] for this representation. See also Pardoux [120, Section 4.1] for a short introduction to those martingale measures and El Karoui and Méléard [51] for a more general framework.

The stochastic equation (A.17) allows one to understand the dynamics of the process. In particular, the role of each parameter in the branching mechanism Ψ is made clear. For the sake of clarity, fix $s = 0$ and denote $X_{0,t}$ by X_t. Before a time atom t of $\mathcal{N}$, an individual u is picked uniformly from the interval of extant individuals $[0, X_{t-}]$ and then reproduces by generating a random mass h of offspring. When h is small, the resulting jump is compensated by a burst of negative drift, akin to modeling natural deaths. The parameter $\gamma \in \mathbb{R}$ drives a deterministic exponential growth/decay and $\sigma \in [0, \infty)$ is the

Feller's diffusion coefficient. The parameter λ, which does not appear explicitely in the equation can be seen as a mass at ∞ for the intensity measure π (hence a jump of infinite size is allowed).

Another interesting aspect of the stochastic equation (A.17), especially with the white-noise term, is that the flow of processes $(X_{s,t}(z), t \geq s, z \geq 0)$ is defined within the same probability space. Furthermore, the properties of the random measures $\widetilde{B}$ and $\mathcal{N}$, as for instance their independence over disjoint intervals, ensure that $X_{s,t}(\cdot)$ is a càdlàg subordinator.

A.2.2 Feller property, infinitesimal generator and martingale problem

Any CSBP is a Feller process in the sense that its semigroup maps the space $C([0,\infty])$, of continuous functions on $[0,\infty]$ (hence with a limit at ∞) to itself. We stress also that $[0,\infty]$ can be endowed with the compact metric $d(x,y) = |e^{-x} - e^{-y}|$, with by convention $e^{-\infty} = 0$.

Let Ψ be Lévy-Khintchine function, denote by L^Ψ the generator of a spectrally positive Lévy process with Laplace exponent Ψ and call $\mathscr{L}^\Psi$ the generator of the CSBP(Ψ). Let $\mathcal{D}$ be the space of functions $\mathcal{D} := \{f \in C^2(0,\infty) :$ the limit $f(\infty) := \lim_{z\to\infty} f(z)$ exists in $\mathbb{R}\}$. One has for any $f \in \mathcal{D}$, $\mathscr{L}^\Psi f(z) := z\mathrm{L}^\Psi f(z)$, namely

$$\mathscr{L}^\Psi f(z) = \frac{\sigma^2}{2} z f''(z) + \gamma z f'(z) + z\int_0^\infty \left(f(z+h) - f(z) - hf'(z)\mathbb{1}_{\{h\leq 1\}}\right)\pi(\mathrm{d}h) + \lambda z(f(\infty) - f(z)). \quad \text{(A.18)}$$

The process X is the unique solution to the following (local) martingale problem:

$$(\mathrm{MP})_X : \qquad \forall f \in \mathcal{D}, \text{ the process } (M_t, t\geq 0) := \left(f(X_t) - \int_0^t \mathscr{L}^\Psi f(X_s)\mathrm{d}s, t\geq 0\right) \text{ is a local martingale.} \quad \text{(A.19)}$$

If the function $f \in \mathcal{D}$ verifies $\sup_{z\in[0,\infty]} |\mathscr{L}^\Psi f(z)| < \infty$, the process $(M_t, t \geq 0)$ is a genuine martingale. This is the case for instance when $\lim_{z\to\infty} z|f(z)+f'(z)+f''(z)| = 0$. In particular, the linear span of the exponential functions, $\mathrm{Vect}\{e_x(\cdot), x > 0\}$, with $e_x(z) := e^{-xz}$ and the convention $0 \times \infty = 0$, lies in the domain of the generator of X. It furthermore forms a core of the latter. Lastly, notice the action of the generator on the exponentials:

$$\mathscr{L}^\Psi e_x(z) = z\mathrm{L}^\Psi e_x(z) = z e_x(z)\Psi(x) = -\Psi(x)\frac{\mathrm{d}}{\mathrm{d}x} e_x(z). \quad \text{(A.20)}$$

A.2.3 Lamperti's time change

A simple construction of the CSBP with mechanism Ψ is provided by time-changing a spectrally positive Lévy process $(Y_t, t \geq 0)$ with Laplace exponent Ψ, that is stopped when reaching 0. If $\lambda > 0$, Y jumps to ∞ at an independent exponential time $\mathbb{e}_\lambda$ with parameter λ. Denote by $\sigma_0 := \inf\{t > 0 : Y_t \leq 0\}$, the first passage time below 0 of Y.

Define the additive functional θ and its right-inverse C by

$$t \mapsto \theta_t := \int_0^{t\wedge\sigma_0} \frac{\mathrm{d}s}{Y_s} \in [0,\infty] \text{ and } t \mapsto C_t := \inf\{u \geq 0 : \theta_u > t\} \in [0,\infty], \quad \text{(A.21)}$$

with the usual convention $\inf\{\emptyset\} = \infty$. The Lamperti time-change of the stopped process $(Y_t, t \geq 0)$ is the process $(X_t, t \geq 0)$ defined by

$$X_t = \begin{cases} Y_{C_t} & 0 \leq t < \theta_\infty := \int_0^{\sigma_0} \frac{\mathrm{d}s}{Y_s}, \\ 0 & t \geq \theta_\infty \text{ and } \sigma_0 < \infty, \\ \infty & t \geq \theta_\infty \text{ and } \sigma_0 = \infty. \end{cases} \tag{A.22}$$

By applying standard techniques of time-change, see Lamperti [96] and Volkonskii [139], and e.g. Ethier-Kurtz's book [56, Theorem 1.4 page 309], the identity $\mathscr{L}^\Psi f(z) = z\mathrm{L}^\Psi f(z)$ for all $f \in \mathcal{D}$ and $z \in (0, \infty)$, it can be shown that this process X is a càdlàg solution to $(\mathrm{MP})_X$ and thus is a CSBP with branching mechanism Ψ. We refer also to Caballero et al. [34] for a different proof.

When the parent Lévy process Y is a subordinator (and thus $-\Psi$ is its Laplace exponent), the CSBP with branching mechanism Ψ has almost surely non-decreasing sample paths and is said to be *immortal* as in the discrete setting.

A.2.4 Long-term behaviors

Grey in [68] and Bingham [25] have studied the long-term behavior of CSBPs. We sum up the classical results of Grey, [68] below. They can also be found in Kyprianou's book [88, Chapter 12] and Li's book [109, Chapter 3]. Recall that a striking feature of the continuous-state space is that the population can become *extinguished*, namely the population size goes towards zero while maintaining a positive value at any time.

Let $\varrho := \inf\{x > 0 : \Psi(x) > 0\} \in [0, \infty]$ be the largest positive zero of Ψ.

Theorem A.22 (Grey, [68]). *Consider $(X_t(z), t \geq 0)$ a CSBP(Ψ) started from z.*

i) For any $z \geq 0$,

$$\mathbb{P}(\lim_{t\to+\infty} X_t(z) = 0) = 1 - \mathbb{P}(\lim_{t\to+\infty} X_t(z) = +\infty) = e^{-z\varrho}$$

ii) The process is almost surely not absorbed at 0 if and only if $\int^{+\infty} \frac{\mathrm{d}u}{|\Psi(u)|} = +\infty$. If $\int^{+\infty} \frac{\mathrm{d}u}{|\Psi(u)|} < \infty$ (Grey's condition), then Ψ is positive near ∞, $\Psi'(\infty) = \infty$ and the following limits exist

$$u_t(\infty) = \lim_{x\to+\infty} u_t(x) \in (0, +\infty) \text{ for any } t \geq 0 \text{ and } \lim_{t\to+\infty}\downarrow u_t(\infty) = \varrho.$$

Moreover for all $t \geq 0$, $\frac{\mathrm{d}}{\mathrm{d}t} u_t(\infty) = -\Psi(u_t(\infty))$ with $u_0(\infty) = +\infty$,

$$\mathbb{P}(X_t(z) = 0) = e^{-zu_t(\infty)} \text{ and } \mathbb{P}(\exists t \geq 0 : X_t(z) = 0) = e^{-z\varrho}.$$

iii) The process is almost surely not absorbed at ∞ if and only if $\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} = \infty$. If $\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} < \infty$ (Dynkin's condition), then Ψ is negative near 0, $\Psi'(0) = -\infty$ and the following limits exist:

$$u_t(0+) = \lim_{x\to 0} u_t(x) \in (0, +\infty) \text{ for any } t \geq 0 \text{ and } \lim_{t\to+\infty}\uparrow u_t(0+) = \varrho.$$

Moreover for all $t \geq 0$, $\frac{\mathrm{d}}{\mathrm{d}t} u_t(0+) = -\Psi(u_t(0+))$ with $u_0(0+) = 0$ and

$$\mathbb{P}(X_t(z) = +\infty) = 1 - e^{-zu_t(0+)} \text{ and } \mathbb{P}(\exists t \geq 0 : X_t(z) = +\infty) = 1 - e^{-z\varrho}.$$

A.2.5 Classification according to the mean or the variation

We classify now the mechanisms Ψ according to their behaviour near 0 and $+\infty$. For any Ψ as in (A.16),

$$\Psi'(0+) = \lim_{u\to 0} \frac{\Psi(u)}{u} = \gamma - \int_1^{+\infty} x\pi(\mathrm{d}x) \in [-\infty, +\infty) \quad \text{(mean)}.$$

Similarly as in the discrete-state space, we distinguish three regimes: supercritical, critical and subcritical. The process is said to be supercritical if $\Psi'(0+) \in [-\infty, 0)$ (in which case $\varrho \in (0, +\infty]$), subcritical if $\Psi'(0+) \in (0, +\infty)$ and critical if $\Psi'(0+) = 0$ (in these last two cases $\varrho = 0$). One has moreover $\mathbb{E}[X_t(z)] = ze^{-\Psi'(0+)t}$ for all $t \geq 0$, this leads to the following classification.

- If $\Psi'(0+) \in (-\infty, 0)$, the process has a finite mean and $\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} = \infty$. Therefore the process does not explode almost-surely and goes to $+\infty$ with probability $1 - e^{-x\varrho}$.
- If $\Psi'(0+) = -\infty$ and $\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} = \infty$ then the process has an infinite mean, does not explode almost-surely and goes to $+\infty$ with probability $1 - e^{-x\varrho}$.
- If $\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} \in (0, \infty)$ (then $\Psi'(0+) = -\infty$), the process explodes with probability $1 - e^{-x\varrho}$ if $\underline{u} < +\infty$, with probability 1 if $\underline{u} = +\infty$.

In a somewhat symmetrical fashion, we introduce now the *variation*. For any Ψ,

$$\Psi'(\infty) := \lim_{u\to+\infty} \frac{\Psi(u)}{u} = +\infty 1_{\{\sigma>0\}} + \gamma + \int_0^1 x\pi(\mathrm{d}x) \in (-\infty, +\infty] \quad \text{(variation)}.$$

- We say that Ψ is of finite variation when $\Psi'(\infty) \in \mathbb{R}$. In this case the CSBP process, and its parent Lévy process, has finite variation sample paths and $\int^\infty \frac{\mathrm{d}u}{|\Psi(u)|} = +\infty$. Therefore the process is persistent (not absorbed at 0 almost-surely) and goes to 0 with probability $e^{-x\varrho}$ ($\varrho = +\infty$ if $\Psi'(\infty) \leq 0$).

  Note that $\Psi'(\infty) \in \mathbb{R}$ if and only if $\sigma = 0$ and $\int_0^1 u\pi(\mathrm{d}u) < +\infty$. In this case (A.16) can be rewritten as

$$\Psi(u) = -\lambda + \Psi'(\infty)u - \int_0^{+\infty} \pi(\mathrm{d}r)(1 - e^{-ur}). \tag{A.23}$$

  The four following facts are equivalent:

  $\Psi'(\infty) \leq 0$; $\Psi(\infty) = -\infty$; $\varrho = \infty$; $-\Psi$ is the Laplace exponent of a subordinator.
- We say that Ψ is of infinite variation when $\Psi'(\infty) = +\infty$. In this case the CSBP process, and its parent Lévy process, has infinite variation sample paths. Furthermore $\Psi(\infty) = \infty$ and :
  - If $\int^{+\infty} \frac{\mathrm{d}u}{\Psi(u)} = +\infty$, then the process is *persistent* (i.e. does not hit 0) and goes to 0 with probability $e^{-x\varrho}$.
  - If $\int^{+\infty} \frac{\mathrm{d}u}{\Psi(u)} < +\infty$ (then $\Psi'(\infty) = +\infty$), the process has infinite variation sample paths and is absorbed at 0 with probability $e^{-x\varrho}$.

Example A.23. 1. The stable CSBP with parameter $\alpha \in (0, 1)$, i.e.

$$\Psi(x) := -dx^\alpha = -\frac{d\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-xh}) \frac{\mathrm{d}h}{h^{1+\alpha}},$$

has finite variation and explodes a.s.

2. The stable CSBP with parameter $\alpha \in (1, 2]$, i.e.

$$\Psi(x) = dx^\alpha = \frac{d\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_0^\infty (e^{-xh} - 1 + xh)\frac{\mathrm{d}h}{h^{1+\alpha}},$$

has infinite variation and gets extinct in finite time a.s.

3. The Neveu's CSBP, i.e. the process whose branching mechanism is

$$\Psi(u) = x \log x = (\gamma - 1)x + \int_0^{+\infty} (e^{-xh} - 1 + xh\mathbb{1}_{\{h \le 1\}})h^{-2}\mathrm{d}h, \tag{A.24}$$

with $\gamma = \int_1^\infty e^{-h}h^{-2}\mathrm{d}h$ (Euler-Maschenori constant), has infinite mean, $\Psi'(0+) = -\infty$, and infinite variation, $\Psi'(\infty) = \infty$. Moreover it neither gets extinct nor explodes.

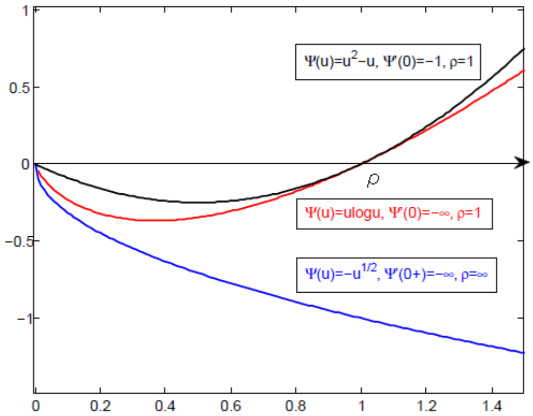


Figure 1 – Supercritical mechanisms

A.2.6 The triplet of the cumulant $u_t(\cdot)$ (Table 1.1)

The branching property (A.14) ensures that $X_t(x)$ is a positive infinitely divisible random variable for all $t \ge 0$. Its log-laplace $u_t(\cdot)$ takes then the Bernstein form: for any $x > 0$

$$u_t(x) = \kappa_t + d_t x + \int_0^\infty (1 - e^{-xr})\ell_t(\mathrm{d}r),$$

with $\kappa_t \ge 0, d_t \ge 0$ and $\int_{(0,\infty)} 1 \wedge r\, \ell_t(\mathrm{d}r) < \infty$. Moreover, one can link the classification above with properties of the triplet as follows:

— $u_t(0+) = \kappa_t > 0 \iff \int_0 \frac{\mathrm{d}u}{|\Psi(u)|} < \infty$,

— $d_t = \lim_{x\to\infty} \frac{u_t(x)}{x} = e^{-\Psi'(\infty)t} > 0 \iff \Psi'(\infty) < \infty$,

— $u_t(\infty) = d_t.\infty + \ell_t((0,\infty)) + \kappa_t < \infty \iff \int^\infty \frac{\mathrm{d}u}{|\Psi(u)|} < \infty$,

— $\frac{\partial}{\partial x}u_t(0) = \infty.\kappa_t + d_t + \int_0^\infty r\ell_t(\mathrm{d}r) < \infty \iff \Psi'(0+) > -\infty$.

A.2.7 Quasi-stationary distribution of CSBPs

Lambert [93] and Li [105] have studied, among other things, the quasi-stationary distribution of subcritical CSBPs conditioned on the non-extinction.

Theorem A.24 (Lambert [93], Li [105]). *In the subcritical case, i.e.* $\Psi'(0+) > 0$, *under Grey's condition* $\int^{\infty} \frac{du}{\Psi(u)} < \infty$, *there exists a probability measure* ν_∞ *over* $(0, \infty)$ *such that for any Borelian set* $A \subset (0, \infty)$ *and any* $z \in (0, \infty)$,

$$\nu_\infty(A) := \lim_{t\to\infty} \mathbb{P}(X_t(z) \in A | X_t(z) > 0).$$

The Laplace transform of ν_∞ *is given by*

$$\int_0^\infty e^{-uz}\nu_\infty(\mathrm{d}z) = 1 - e^{-\Psi'(0^+)\int_u^\infty \frac{\mathrm{d}x}{\Psi(x)}} \text{ for any } u \geq 0. \tag{A.25}$$

B Extremal processes

Extremal processes were encountered in Chapter 1. We gather here some fundamental facts on them. We refer for instance to Chapter 4, Section 3 in Resnick's book [126]. We denote by $x \vee y$ and $x \wedge y$ respectively the maximum and the minimum value between x and y. Let F be a probability distribution function with a given support $(s_l, s_o) \subset \overline{\mathbb{R}}$. For all $x \geq 0$, set $Q(x) := -\log F(x)$. A real-valued process $(M_x, x \geq 0)$ is an extremal-F process if for any $0 \leq x_1 \leq ... \leq x_n$ and $(z_1, ..., z_n) \in \mathbb{R}^n$,

$$\mathbb{P}(M_{x_1} \leq z_1, M_{x_2} \leq z_2, ..., M_{x_n} \leq z_n) = e^{-x_1 Q(z_1')}e^{-(x_2-x_1)Q(z_2')}...e^{-(x_n-x_{n-1})Q(z_n')} \tag{B.26}$$

where $z_i' = \wedge_{k=i}^n z_k$ for all $i \geq 1$. Any extremal-F process $(M_x, x \geq 0)$ has the following properties:

i) $(M_x, x \geq 0)$ is stochastically continuous.

ii) $(M_x, x \geq 0)$ has a càdlàg version.

iii) $(M_x, x \geq 0)$ has a version with non-decreasing paths such that $\lim_{x\to+\infty} M_x = s_o$ and $\lim_{x\to 0} M_x = s_l$ almost-surely.

iv) $(M_x, x \geq 0)$ is a Markov process with for $x > 0, y > 0$,

$$\mathbb{P}(M_{x+y} \leq z \,| M_x = v) = \begin{cases} e^{-yQ(z)} & \text{if } z \geq v \\ 0 & \text{if } z < v. \end{cases} \tag{B.27}$$

The parameter of the exponential holding time in state x is thus $Q(x)$, and the process jumps from x to $(x, z]$ with probability $1 - \frac{Q(z)}{Q(x)}$. The only possible instantaneous state is s_l and it is instantaneous if and only if $F(s_l) = 0$.

A constructive approach of extremal processes is given by the records of a Poisson point process. Let μ be a σ-finite measure on $(0, +\infty)$, and consider a Poisson point process $\mathcal{P} = \sum_{i\in I}\delta_{(x_i, Z_i)}$ with intensity $dx \otimes \mu$. The process $(M_x, x \geq 0)$ defined by $M_x := \sup_{x_i \leq x} Z_i$ is a càdlàg extremal-F process with for all $z \in \mathbb{R}$, $F(z) = \exp(-\overline{\mu}(z))$ where $\overline{\mu}(z) = \mu(z, +\infty)$. We highlight that the state 0 is instantaneous if the intensity measure μ is infinite. The positive extremal processes will play an important role in the sequel. They correspond to the records of Poisson point processes over $\mathbb{R}_+ \times \mathbb{R}_+$.

An interesting feature of extremal processes lies in their *max-infinite divisibility*. Namely, for any integer m,

$$(M_x, x \geq 0) \overset{d}{=} (\max_{i\in[|1,m|]} M_x^i, x \geq 0) \tag{B.28}$$

where $(M^i_x, x \geq 0)_{i\in[|1,m|]}$ are i.i.d extremal-$F^{1/m}$ processes. Indeed,

$$\mathbb{P}(\max_{i\in[|1,m|]} M^i_{x_1} \leq z_1, ..., \max_{i\in[|1,m|]} M^i_{x_n} \leq z_n) = \left(F(z_1')^{\frac{x_1}{m}} F(z_2')^{\frac{x_2-x_1}{m}} ... F(z_n')^{\frac{x_n-x_{n-1}}{m}}\right)^m$$
$$= \mathbb{P}(M_{x_1} \leq z_1, M_{x_2} \leq z_2, ..., M_{x_n} \leq z_n).$$

C One-dimensional diffusions and Feller's boundary classification

C.1 Boundaries terminology

Given a càdlàg strong Markov process valued in $[0, \infty]$, we say that the boundary 0 (respectively ∞) is *accessible* if the process hits 0 (respectively ∞) with positive probability. Otherwise, we say that the boundary is *inaccessible*.

When a boundary is inaccessible, it can be either an entrance or a natural boundary. In the *entrance* case, although it cannot hit the boundary, the process can be started from it, that is to say, if the process is initially at the boundary, then it will leave it at some future time. In the *natural* case, the process neither can leave nor hit the boundary.

When a boundary is accessible, it can be either an exit or a regular boundary. In the *exit* case, the process cannot leave the boundary and thus stays at it after it has reached it. In the *regular* case, the process can leave the boundary (in various ways) if it is not stopped upon reaching it. We shall distinguish two cases for a regular boundary. The boundary will be called regular *reflecting* when the time spent by the process at the boundary has a zero Lebesgue measure. When the process is stopped at a regular boundary, the boundary is said to be regular *absorbing*. We stress that in this document, all processes under consideration (apart from the CBIs and ESNs briefly mentioned in the last section of Chapter 1) leave in a *continuous* way a boundary that is non-absorbing (regular reflecting or entrance boundary).

Lastly, a regular boundary is *regular for itself* if the process returns immediately after having left it. For a broad class of processes, this entails the existence of a local time at the boundary and the decomposition of the paths into excursions away from the boundary along a Poisson point process. We refer the reader for instance to Blumenthal and Getoor's book [31, Chapter V, Section 3] and Bertoin [16, Chapter 4, Section 2] for two different constructions of the local time.

C.2 One-dimensional diffusions on $[0, \infty]$

C.2.1 Stochastic differential equations and martingale problem

Let σ and μ be continuous functions on $[0, \infty)$ and assume σ strictly positive on $(0, \infty)$. Consider the following SDE

$$\mathrm{d}U_t = \sigma(U_t)\mathrm{d}B_t + \mu(U_t)\mathrm{d}t,\ U_0 = x \in (0, \infty), \tag{C.29}$$

for some Brownian motion $(B_t, t \geq 0)$. There exists a unique weak solution $(U_t, t \geq 0)$ that is stopped at $S := \inf\{t > 0 : U_t \notin (0, \infty)\}$, see for instance Karatzas and Shreve's book [82, Section 5.5-C, pages 343-344]. We call it the minimal solution. It has a version with continuous sample paths and for any $t \geq S$, $U_t = 0$ if $S = \tau_0 := \inf\{t > 0 : U_t = 0\} < \infty$ and $U_t = \infty$ if $S = \tau_\infty := \inf\{t > 0 : U_t = \infty\} < \infty$.

Moreover, $(U_t, t \geq 0)$ is the minimal solution of Equation (C.29) if and only if it has absorbing boundaries and it satisfies the following martingale problem $(\mathrm{MP})_U$: for any $f \in C_c^2(0,\infty)$, the space of twice differentiable functions with compact support, the process

$$\left(f(U_t) - \int_0^t \mathscr{A} f(U_s)\mathrm{d}s, t \geq 0\right) \text{ is a martingale,} \tag{C.30}$$

where $\mathscr{A}$ is called the generator and takes the form

$$\mathscr{A} f(x) = \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x), \ \forall f \in C^2(0,\infty).$$

We refer e.g. to Durrett's book [49, Section 6.1] for a study of $(\mathrm{MP})_U$. The minimal solution does not take into account the behaviors at the boundaries, in the sense that as soon as one boundary is accessible, it is absorbing for the minimal diffusion.

C.2.2 First passage times

Denote by T_y the first hitting time of $y \in [0,\infty]$ of a diffusion $(U_t, t \geq 0)$ and set $\mathscr{A}$ as its generator. From the general theory of one-dimensional diffusions, see e.g. Mandel [113, Chapter V, Section 1] and Borodin and Salminen [32, Chapter II, Section 10], the Laplace transform of T_y is expressed, for any $\theta > 0$, as

$$\mathbb{E}_x[e^{-\theta T_y}] = \begin{cases} \frac{h_\theta^+(x)}{h_\theta^+(y)}, & x \leq y, \\ \frac{h_\theta^-(x)}{h_\theta^-(y)}, & x \geq y, \end{cases} \tag{C.31}$$

where the functions h_θ^- and h_θ^+ are C^2 and respectively decreasing and increasing solutions to the equation

$$\mathscr{A} h(x) = \theta h(x), \text{ for all } x \in (0,\infty). \tag{C.32}$$

Other solutions to the martingale problem $(\mathrm{MP})_U$ than the minimal one may exist and all the different behaviors explained in Section C.1 may possibly happen at a non natural boundary for some functions σ, μ. In a parallel way, if $y \in \{0,\infty\}$ is a regular boundary, they are several solutions to the equation (C.32) and one has to impose a boundary condition on the solution [4] according to the condition at the boundary. We stress that several processes with generator $\mathscr{G}$ on $(0,\infty)$ do exist when a boundary is regular.

In order to classify the boundaries, we need the scale function and speed measure.

C.2.3 Scale function and speed measure

Let x_0, y_0 be arbitrary fixed points in $(0,\infty)$. Set $s_U(x) := \exp\left(-\int_{x_0}^x \frac{2\mu(u)}{\sigma^2(u)}\mathrm{d}u\right)$ and

$$S_U(y) := \int_{y_0}^y s_U(x)\mathrm{d}x = \int_{y_0}^y \exp\left(-\int_{x_0}^x \frac{2\mu(u)}{\sigma^2(u)}\mathrm{d}u\right)\mathrm{d}x. \tag{C.33}$$

We call S_U the scale function and shall also denote by S_U the Stieltjes measure associated to S_U. Let $m_U(x) := \frac{1}{\sigma^2(x)s_U(x)}$ for any $x \in (0,\infty)$ and

$$M_U(y) := \int_{y_0}^y m_U(x)\mathrm{d}x = \int_{y_0}^y \frac{1}{\sigma^2(x)} \exp\left(\int_{x_0}^x \frac{2\mu(u)}{\sigma^2(u)}\mathrm{d}u\right)\mathrm{d}x. \tag{C.34}$$

We denote also by M_U the associated Stieltjes measure, this is the so-called speed measure.

4. the condition is on h_θ^+ for $y = 0$, on h_θ^- for $y = \infty$

An important fact is that M_U and S_U characterize completely the law of the minimal diffusion U. Moreover, the one-dimensional law of U admits a density with respect to the speed measure M_U. In our case the latter will always be absolutely continuous and in particular for any $t > 0$, the law of U_t has no atom in $(0, \infty)$. We refer e.g. to Rogers and Williams [128, Theorem 50.11, Chapter V].

C.2.4 Feller's conditions

The classification of boundaries of one-dimensional diffusions is completely understood. We refer for instance to Karlin and Taylor's book [83, Chapter 15, Section 6]. We briefly recall the integral tests that are specifying the behavior of a diffusion at its boundaries.

Let $x_0 \in (0, \infty)$ be an arbitrary point. For any $l \in [0, \infty]$, define the integral tests I_U and J_U by

$$I_U(l) := \int_l^{x_0} S_U[l, x] \mathrm{d}M_U(x) \text{ and } J_U(l) := \int_l^{x_0} S_U[u, x_0] \mathrm{d}M_U(u). \tag{C.35}$$

The boundary 0 (respectively ∞) is accessible if and only if $I_U(0) < \infty$ (respectively $I_U(\infty) < \infty$).

The following analytical classification of boundaries can be found for instance in [83, Table 6.2, page 234].

Feller's conditions	Boundary of U
$S_U(0, x] < \infty$ and $M_U(0, x] < \infty$	0 regular
$S_U(0, x] = \infty$ and $J_U(0) < \infty$	0 entrance
$M_U(0, x] = \infty$ and $I_U(0) < \infty$	0 exit
$I_U(0) = \infty$ and $J_U(0) = \infty$	0 natural

Table 3 – Boundaries of U.

By replacing everywhere 0 by ∞ in Table 3, we get the classification for the boundary ∞. In the regular case, an extra information on the behavior at the boundary is needed to completely understand the process, see e.g. Borodin and Salminen [32] for the complete classification. We only consider the two extreme possibilities namely reflection and absorption (sticky behavior interpolates between the two and is not considered anywhere in the document).

When a boundary is natural or is absorbing (i.e. exit or regular absorbing), we say that the boundary is attracting if the process has a positive probability to converge towards it. When moreover both boundaries are attracting, the process will converge towards one of them almost surely. We have the following classification, see e.g. [82, Proposition 5.22, page 345]:

Conditions	Boundary of U
$S_U(0, x] < \infty$	0 attracting
$S_U[x, \infty) < \infty$	∞ attracting

Table 4 – Attracting boundaries of U.

C.2.5 Feller's construction of a diffusion with a regular reflecting boundary

The previous paragraph classifies the boundaries without explaining how a diffusion can be constructed with a given behavior at the boundary. Only the case regular *reflecting* really needs to be explained. Let U be a positive diffusion. Assume that its boundary 0 is regular, that is $S_U(0,x] < \infty$ and $M_U(0,x] < \infty$. The diffusion U solution to (C.29) with 0 regular reflecting can be constructed as follows, see e.g. [49, Section 6.5]. Let S_U be the scale function normalized such that $S_U(0) = 0$. Consider the process absorbed after its first hitting time of 0, call it $(U_t^a, t \geq 0)$. The diffusion $(S_U(U_t^a), t \geq 0)$ is in natural scale with speed density measure $1/h$, defined by

$$h(y) := S'_U(S_U^{-1}(y))^2 S_U^{-1}(y) \text{ for } y \in [0, \infty),$$

see for instance [49, Equation (1.5), Section 6.1, page 212]. Extend h on $\mathbb{R}$ by $h(-y) = h(y)$ for all y, let $(X_t, t \geq 0)$ be the diffusion on $\mathbb{R}$ in natural scale (i.e. $S_X(x) = x$) with speed density measure $f(y) = 1/h(|y|)$ for all $y \in \mathbb{R}$. The assumption $M_U(0,x] < \infty$ actually ensures that the diffusion X started from 0, hits *almost-surely* a point $y \neq 0$, hence X is not absorbed at 0. Moreover, the zero-set of X has zero Lebesgue measure (its speed measure does not give mass to 0). Finally, the diffusion U defined by $U_t := S_U^{-1}(|X_t|)$ for all $t \geq 0$, has its boundary 0 regular reflecting and extends $(U_t^a, t \geq 0)$.

It is worth mentioning that this construction does not appeal to the local time. The latter is of course hidden in the reflected time-changed Brownian motion X.

D H-duality of processes

Let E and F be two nice spaces and H be a positive or bounded measurable function defined on $E \times F$. Two processes $X := (X_t, t \geq 0)$ and $Y := (Y_t, t \geq 0)$ valued respectively in E and F are said to be an H-duality relationship *at the level of their semigroups* if for any $t \geq 0$ and $x \in E$, $y \in F$:

$$\mathbb{E}_x^X[H(X_t, y)] = \mathbb{E}_y^Y[H(x, Y_t)], \tag{D.36}$$

where $\mathbb{E}_x^X$ denotes the expectation operator with respect to $\mathbb{P}_x^X$ the law of the process X started from x, and similarly for the process Y. We refer to Jansen and Kurt's survey about the notion(s) of duality [79].

The notion of H-duality (also called Liggett's duality by some authors) plays also a predominant role in the literature of interacting particle systems. Another famous application of H-duality is to help proving uniqueness of martingale problems in superprocesses theory. This is the so-called duality method, we refer the reader to Etheridge's book [53] and Ethier-Kurtz's book [56].

Apart from the case $H(x,y) = \mathbb{1}_{\{x<y\}}$, for which Siegmund [135] found a necessary and sufficient condition for a process X to admit an H-dual, there is no general treatment of H-duality theory. Processes in Laplace duality, $H(x,y) = e^{-xy}$, for instance do not seem to be known in their full generality, see however the characterization of CSBPs with collisions and the comments in Section 4.2.

D.1 Basic coupling of independent processes in H-duality

We say that H is law-determining if for any random variable X valued in E, the map $y \mapsto \mathbb{E}(H(X,y))$ defined on F characterizes the law of X. This the case for the three functions:

1. **Laplace duality**: $\forall x, z \in (0,\infty)$, $H(x,z) := e^{-xz} =: e_x(z) = e_z(x)$,
2. **Moment duality**: $\forall x \in (0,1)$, $n \in \mathbb{N}$, $H(x,n) := x^n =: f_x(n) =: f_n(x)$,
3. **Siegmund duality**: $\forall x, y \in (0,\infty)$, $H(x,y) := \mathbb{1}_{\{x<y\}} = 1 - \mathbb{1}_{\{x\geq y\}}$.

If X and Y are two processes satisfying (D.36) and H is law-determining then if Y is a Markov process, so is X. Moreover if we take them independent, then one also plainly check from the Markov property that the process

$$s \in [0,t] \mapsto H(X_s, Y_{t-s}) \tag{D.37}$$

is a martingale. Heuristically we see here that a duality relation relates the evolution of X in its natural time to the time-reversal of its dual Y.

D.2 From generators duality to that of semigroups

Denote by $\mathscr{L}^X$ and $\mathscr{L}^Y$ the infinitesimal generators of X and Y and set $H_y(x) := H(x,y) =: H^x(y)$. We say that the processes are in pre-duality or in H-duality *at the level of their generators* if

$$\mathscr{L}^X H_y(x) = \mathscr{L}^Y H^x(y) \tag{D.38}$$

provided of course that H is smooth enough for the generators above to make sense when they are acting on it [5]. The duality relationships at the two levels (D.36) and (D.38) are generally not equivalent. If one of the boundaries of X is regular, different boundary conditions (leading to different processes) could be imposed on it without breaking the identity (D.38) outside boundaries. This explains why moving from (D.38) to (D.36) is not straightforward in general (at least without specifying the behavior at the boundaries). By assuming that the functions $y \mapsto H(x,y)$, $y \mapsto \mathbb{E}_x(H(X_t,y))$ belong to the domain of the infinitesimal generator of Y, as well as $x \mapsto H(x,y)$, $x \mapsto \mathbb{E}_y(H(x,Y_t))$ to that of X, one can indeed show directly from (D.38) and Dynkin's formula that (D.37) is a martingale. Evaluating its expectation at time $s=0$ and $s=t$ yields (D.36).

The assumptions made above is however very restrictive as for most processes (even for most Markov chains), we do not know how to describe the domain nor how to setup the boundary conditions. Notice that even if we knew a core for X, it would remain to see why for instance the map $x \mapsto \mathbb{E}_y(H(x,Y_t))$ is in it.

The most well-known results allowing to go from (D.38) to (D.36) are the sufficient conditions, given by Ethier and Kurtz in their book [56, Theorem 4.11, Corollaries 4.13-15] (with possibly requiring the processes to be stopped before certain random times). They relie on the idea to verify that (D.37) is indeed a martingale (or a local one) for X and Y independent.

The condition reads as follows: assume $G(x,y) := \mathscr{L}^X H_y(x) = \mathscr{L}^Y H_x(y)$. If for any fixed time T, the random variables,

$$\Gamma_H := \sup_{s,t\leq T} H(X_s,Y_t) \text{ and } \Gamma_G := \sup_{s,t\leq T} G(X_s,Y_t),$$

are integrable, then (D.36) holds. For instance in the setting of LCSBPs with finite mean as studied in Chapter 3.2, Γ_H is bounded, since H is bounded, and only Γ_G has to be studied. Its integrability can be checked with some work.

5. this prevents for instance a direct study of Siegmund duality from this approach

To conclude, in the paper [42], Cox and Rösler explain how by using an H-duality relation one can relate the entrance laws of one Markov process to the *exit laws* of the other (see e.g. Dynkin's book [50, page 278] for the notion of exit law). In some sense, this is what is happening behind the use of duality in Part II and Part III. Notice that the introduction of a second duality relationship, as what we do in Part II with the processes Z, U, V, allows us to relate entrance laws of the process Z to those of its bidual process V and so to skip the exit laws theory.

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