Short-Course AMSS (C. Foucart)

Talk 3: Continuous-state branching processes with competition:

logistic CSBPs

Laplace duality and reflection at ∞.

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Introduction

Imagine a random continuous population with the following dynamics:

- Each individual reproduces independently, with the same law (as in a continuous-state branching process (CSBP))
- At constant rate, two individuals are picked in the population, and one kills the other (quadratic competition).

The total size of the population, say $(Z_t, t \geq 0)$, is called **logistic continuous-state branching process** (Lambert 2005). Formally,

$$\mathrm{d}Z_t = \text{CSBP dynamics} - \frac{c}{2} Z_t^2 \mathrm{d}t. \tag{1}$$

This is a random analogue of the **logistic function** introduced by Verhulst (1844) and solving:

$$\mathrm{d}z_t = \gamma z_t \mathrm{d}t - \frac{c}{2} z_t^2 \mathrm{d}t. \tag{2}$$

One can solve (2) explicitly. There is an equilibrium at $\frac{2\gamma}{c}$ and it can be started from ∞.

⚠ Competition destroys the branching property. The aim is to study these processes with *general branching mechanisms, including those for which CSBPs explode in finite time.*

Questions

1. *Are there strong enough reproduction laws to face the competition and explosion to occur? (∞* **accessible***.)*
2. *If the process does not explode, is it possible to start it from infinity ? (∞* **entrance***.)*
3. *If the process explodes, can we extend it after its first explosion time continuously or not? (∞* **regular** *or* **exit***.)*
4. ...

- (1) will be solved by arguments of "**time-change**",
- (2) and (3) by "**duality**". We will find NAS conditions for ∞ to be regular and build an extended process $(Z_t, t \geq 0)$ with ∞ regular reflecting, namely s.t.

$$\lambda(\{t > 0, Z_t = \infty\}) = 0 \text{ a.s.}$$

Feller's boundary classification

Consider a process valued in an interval (a, b) with $a < b \in [0, \infty]$,

- the boundary b is **accessible** if the process enters into b with positive probability. If b is accessible, then
  - when the process cannot get out from b, the boundary b is said to be an **exit**
    or
  - when the process can get out from b, the boundary b is called a **regular** boundary.
- If the boundary b is **inaccessible**, then
  - when the process cannot get out from b, the boundary b is said to be **natural**
    or
  - when the process can get out from b, the boundary b is said to be an **entrance**.

In the case of a diffusion, integral tests for each possible boundary are known in terms of the scale function and speed measure, (Feller (1954)).

When a boundary b is regular, the process after hitting b can be extended in several ways, for instance:

- b is regular absorbing; the process stays at b.
- b is regular reflecting, the process leaves b instantaneously and does not spend any positive Lebesgue time on it.

Minimal Logistic CSBPs: definition

Recall the form of a branching mechanism Ψ

$$\Psi(z) = -\lambda + \frac{\sigma^2}{2}z^2 + \gamma z + \int_0^{+\infty} \left(e^{-zx} - 1 + zx\mathbb{1}_{\{x\leq 1\}}\right)\pi(\mathrm{d}x)$$

and $\mathcal{L}^{\mathrm{CSBP}}$ the generator of the CSBP(Ψ), to incorporate quadratic competition, one sets

$$\mathcal{L}f(z) := \mathcal{L}^{\mathrm{CSBP}}f(z) - \frac{c}{2}z^2 f'(z).$$

Definition

A **minimal** *logistic continuous-state branching process is a càdlàg Markov process* $(Z_t^{min}, t \geq 0)$ *on* $[0,\infty]$ *with* 0 *and* ∞ **absorbing**, *satisfying : For any function* $f \in C_c^2((0,\infty))$, *the process*

$$t \mapsto f\left(Z_{t\wedge\zeta}^{min}\right) - \int_0^t \mathcal{L}f\left(Z_{s\wedge\zeta}^{min}\right)\mathrm{d}s \qquad \textbf{(MP)}$$

is a martingale under each $\mathbb{P}_z$, *with* $\zeta := \inf\{t \geq 0; Z_t \notin (0,\infty)\}$.

Existence/uniqueness & explosion

Theorem

There exists a unique minimal logistic CSBP.

Theorem (Accessibility of ∞)

Assume $c > 0$. The boundary ∞ is accessible for $(Z_t^{min}, t \geq 0)$ if and only if

$$\mathcal{E} := \int_0^\theta \frac{1}{x} \exp\left(\frac{2}{c}\int_x^\theta \frac{\Psi(u)}{u}\,\mathrm{d}u\right)\mathrm{d}x < \infty,$$

for some arbitrary $\theta > 0$.

Remark

If $\lambda > 0$ then $\mathcal{E} \propto \int_0 x^{\frac{2\lambda}{c}-1}\mathrm{d}x < \infty$. ⚠ $\lambda > 0$ is not necessary for having $\mathcal{E} < \infty$.

Elements of proof: Existence and Explosion

Existence: time change an OU process (Lambert 05)

Let $(R_t, t \geq 0)$ be an Ornstein-Uhlenbeck type process defined by

$$R_t = z + Y_t - \frac{c}{2}\int_0^t R_s \mathrm{d}s$$

where $(Y_t, t \geq 0)$ is a sp Lévy process with Laplace exponent Ψ. Let $t \mapsto C_t := \inf\{u \geq 0; \theta_u > t\} \in [0, \infty]$ be the right-inverse of

$$\theta_t := \int_0^{t \wedge \sigma_0} \frac{\mathrm{d}s}{R_s}$$

where $\sigma_0 := \inf\{t \geq 0, R_t < 0\}$ and set

$$Z_t^{\min} = \begin{cases} R_{C_t} & 0 \leq t < \theta_\infty \\ 0 & t \geq \theta_\infty \text{ and } \sigma_0 < \infty \\ \infty & t \geq \theta_\infty \text{ and } \sigma_0 = \infty. \end{cases}$$

$(Z_t^{\min}, t \geq 0)$ is a minimal logistic CSBP (i.e. solves **MP**).

Explosion criterion

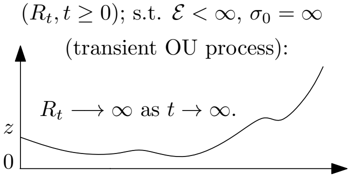
The process $(Z_t^{\min}, t \geq 0)$ hits ∞ if and only if $\sigma_0 = \infty$ and

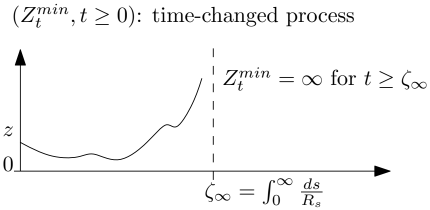
$$\zeta_\infty = \theta_\infty = \int_0^\infty \frac{\mathrm{d}s}{R_s} < \infty.$$

Shiga (PTRF 90) shows that $(R_s, s \geq 0)$ is recurrent if $\mathcal{E} = \infty$ and transient if $\mathcal{E} < \infty$:

- if $(R_s, s \geq 0)$ is recurrent then $\int_0^\infty \frac{\mathrm{d}s}{R_s} = \infty$ on $\{\sigma_0 = \infty\}$.
- if $(R_s, s \geq 0)$ is transient, one will show that on $\{\sigma_0 = \infty\}$

$$\int_0^\infty \frac{\mathrm{d}s}{R_s} < \infty \text{ a.s..}$$





Elements of proof: transience of $R \implies$ explosion of $Z^{\min}$

The Laplace transform of R is given by

$$\mathbb{E}_z(e^{-\theta R_s}) = \exp\left(-\theta e^{-\frac{c}{2}s} z + \int_0^s \Psi(e^{-\frac{c}{2}u}\theta)\mathrm{d}u\right),$$

see e.g. Sato's book. Let $b > 0$. By Tonelli, one has

$$\int_0^\infty \mathbb{E}_z\left(\frac{1 - e^{-bR_s}}{R_s}, \sigma_0 = \infty\right)\mathrm{d}s = \int_0^b \int_0^\infty \mathbb{E}_z(e^{-\theta R_s}, \sigma_0 = \infty)\mathrm{d}s\mathrm{d}\theta$$
$$\leq \int_0^b \int_0^\infty \mathbb{E}_z(e^{-\theta R_s})\mathrm{d}s\mathrm{d}\theta = \frac{2}{c}\int_0^b \mathrm{d}\theta \int_0^\theta \frac{\mathrm{d}x}{x} e^{-xz + \int_x^\theta \frac{2\Psi(v)}{cv}\mathrm{d}v}\mathrm{d}x.$$

The upper bound is finite as soon as $\mathcal{E} = \int_0^\theta \frac{1}{x} e^{\int_x^\theta \frac{2\Psi(v)}{cv}\mathrm{d}v}\mathrm{d}x$ is finite.

Thus if $\mathcal{E} < \infty$,

$$\mathbb{E}_z\left(\int_0^\infty \frac{1-e^{-bR_s}}{R_s}\mathrm{d}s, \sigma_0 = \infty\right) < \infty.$$

We deduce then that on the event $\{\sigma_0 = \infty\}$,

$$\int_0^\infty \frac{1-e^{-bR_s}}{R_s}\mathrm{d}s < \infty \text{ a.s.}$$

Since $\mathcal{E} < \infty$, $R_s \underset{s\to\infty}{\longrightarrow} \infty$ a.s on the event $\{\sigma_0 = \infty\}$ and $\frac{1-e^{-bR_s}}{R_s} \underset{s\to\infty}{\sim} \frac{1}{R_s}$ a.s. Therefore

$$\mathbb{P}_z\left(\int_0^\infty \frac{\mathrm{d}s}{R_s} < \infty | \sigma_0 = \infty\right) = 1,$$

and the process $(Z_t^{\min}, t \geq 0)$ explodes.

Remark

There is no transience in LCSBPs, in the sense that the only way to converge towards ∞ is to hit it.

Generalized Feller diffusions and the key lemma

For all $x \in [0, \infty[$ and $z \in [0, \infty[$, let $e_x(z) := e^{-xz} = e_z(x)$, then

Lemma (Laplace's duality of generators)

For any $x, z \in (0, \infty)$

$$\mathcal{L}e_x(z) = \mathcal{A}e_z(x) \text{ with } \mathcal{A}f(x) = \frac{c}{2}xf''(x) - \Psi(x)f'(x).$$

Proof.

$$\mathcal{L}e_x(z) = \Psi(x)ze_x(z) + \frac{c}{2}xz^2e_x(z) = -\Psi(x)\frac{\partial e_z(x)}{\partial x} + \frac{c}{2}x\frac{\partial^2 e_z(x)}{\partial x^2}. \qquad \square$$

We call Ψ-generalized Feller diffusion, a diffusion with generator $\mathcal{A}$. Ψ is locally Lipschitz on $(0, \infty)$ thus $\exists!$ strong solution to

$$\mathrm{d}U_t = \sqrt{cU_t}\mathrm{d}B_t - \Psi(U_t)\mathrm{d}t,$$

up to $\tau := \inf\{t > 0, U_t \notin (0, \infty)\}$. ⚠ 0 can be exit, regular or entrance and there is **not** a unique semi-group associated to $\mathcal{A}$ (nor to $\mathcal{L}$).

The possible behavior at the boundaries are as follows

Condition	Boundary of U	Boundary of " Z "
$\mathcal{E} = \infty$	0 exit	∞ entrance
$\mathcal{E} < \infty$ and $0 \leq \frac{2\lambda}{c} < 1$	0 regular (absorbing)	∞ regular (reflecting)
$\frac{2\lambda}{c} \geq 1$	0 entrance	∞ exit
$\int^{\infty} \frac{1}{\Psi} < \infty$	∞ entrance	0 exit
$\int^{\infty} \frac{1}{\Psi} = \infty$	∞ natural	0 natural

Table: Boundaries of U and boundaries of Z

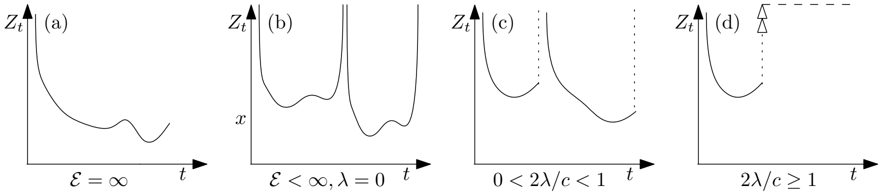


Figure: schematic representation of the four behaviors at ∞.

Infinity as an Entrance Boundary: $\mathcal{E} = \infty$

In the sequel, we say that a process $(Z_t, t \geq 0)$ extends the minimal process if $(Z_{t\wedge\zeta_\infty}, t \geq 0) \overset{\mathcal{L}}{=} (Z_t^{\min}, t \geq 0)$ under $\mathbb{P}_z$ for any $z \in [0, \infty)$.

Theorem (Infinity as entrance boundary)

Assume $\mathcal{E} = \infty$ then 0 is an **exit** *of $(U_t, t \geq 0)$ and $(Z_t^{min}, t \geq 0)$ can be extended to a Feller process $(Z_t, t \geq 0)$ with ∞ as an* **entrance boundary**, *such that for all $t \geq 0$, all $z \in [0, \infty]$, all $x \in [0, \infty)$*

$$\mathbb{E}_z(e^{-xZ_t}) = \mathbb{E}_x(e^{-zU_t})$$

in particular for $z = \infty$,

$$\mathbb{E}_\infty(e^{-xZ_t}) = \mathbb{P}_x(U_t = 0) > 0.$$

Example

Consider $\alpha \in (0,2]$, $\alpha \neq 1$ and $\Psi(z) = (\alpha - 1)z^{\alpha}$, then $\mathcal{E} = \infty$ and ∞ is an **entrance** boundary. For any $t \geq 0$, $z \in [0,\infty]$ and $x \in [0,\infty[$

$$\mathbb{E}_z(e^{-xZ_t}) = \mathbb{E}_x(e^{-zU_t}) \text{ with } \mathrm{d}U_t = \sqrt{cU_t}\mathrm{d}B_t + (1-\alpha)U_t^{\alpha}\mathrm{d}t,$$

the boundary 0 of $(U_t, t \geq 0)$ is an **exit**.

Note that when $\alpha \in (0,1)$, the CSBP without competition explodes, so that here competition prevents explosion.

Infinity as regular reflecting boundary: $\mathcal{E} < \infty$ & $\frac{2\lambda}{c} < 1$

Given Ψ and $k \geq 1$, define $\pi_k = \pi_{|]0,k[} + (\bar{\pi}(k) + \lambda)\delta_k$ and a branching mechanism Ψ_k by

$$\Psi_k(z) := \frac{\sigma^2}{2} z^2 + \gamma z + \int_0^\infty \left(e^{-zx} - 1 + zx \mathbb{1}_{x \in (0,1)}\right) \pi_k(\mathrm{d}x).$$

Call $Z^{(k)}$ the càdlàg logistic CSBP with mechanism Ψ_k and ∞ as entrance boundary.

Theorem (Infinity as regular reflecting boundary)

Assume $\mathcal{E} < \infty$ and $0 \leq \frac{2\lambda}{c} < 1$, then $Z^{(k)} \Longrightarrow Z$ where Z is an extension of Z^{min}, with ∞ **regular reflecting***, and for all $t \geq 0$, all $z \in [0, \infty]$ and $x \in [0, \infty)$,*

$$\mathbb{E}_z(e^{-xZ_t}) = \mathbb{E}_x(e^{-zU^{\mathrm{a}}_t})$$

where $(U^{\mathrm{a}}_t, t \geq 0)$ is solution to $(\star)$ with 0 **regular absorbing**.

Infinity as Exit Boundary: $\frac{2\lambda}{c} \geq 1$

Proposition

If $\mathcal{E} < \infty$ *and* $\frac{2\lambda}{c} < 1$ *then* ∞ *is regular for itself, namely* $S_\infty := \inf\{t > 0, Z_t = \infty\}$ *is such that* $\mathbb{P}_\infty(S_\infty = 0) = 1$.

In particular, there is a local time at ∞. Assume now $\frac{2\lambda}{c} \geq 1$, recall $(Z^{(k)}, k \geq 1)$.

Theorem (Infinity as exit)

Assume $\frac{2\lambda}{c} \geq 1$ *then* 0 *is an* **entrance** *for* $(U_t, t \geq 0)$, *and*

$$Z^{(k)} \Longrightarrow Z$$

where Z *is an extension of* Z^{min}, *with* ∞ **exit** *and for all* $t \geq 0$, *all* $z \in [0, \infty]$ *and* $x \in (0, \infty)$,

$$\mathbb{E}_z(e^{-xZ_t}) = \mathbb{E}_x(e^{-zU_t}).$$

Example ($\rightarrow$ *Fast-fragmentation-coalescence* process, Kyprianou et al. AoP17)

Let $\lambda > 0$ and $\pi \equiv 0$ in order that $\Psi(x) = -\lambda$ for all $x \geq 0$.

- If $\frac{2\lambda}{c} < 1$ then ∞ is **regular reflecting** and $\mathbb{E}_z(e^{-xZ_t}) = \mathbb{E}_x(e^{-zU^0_t})$ with $\mathrm{d}U^0_t = \sqrt{cU^0_t}\mathrm{d}B_t + \lambda \mathrm{d}t$ and 0 **regular absorbing**.
- If $\frac{2\lambda}{c} \geq 1$ then ∞ is an **exit** and $\mathbb{E}_z(e^{-xZ_t}) = \mathbb{E}_x(e^{-zU_t})$ with $\mathrm{d}U_t = \sqrt{cU_t}\mathrm{d}B_t + \lambda \mathrm{d}t$, and 0 is an **entrance**.

Example with **continuous explosion**

Example

Consider $\alpha > 0$, $\beta > 0$ and set $\pi(\mathrm{d}u) = \frac{\alpha}{u(\log u)^2}\mathbb{1}_{\{u \geq 2\}}\mathrm{d}u$.

- If $\frac{2\alpha}{c} \leq 1$ then $\mathcal{E} = \infty$ and ∞ is an **entrance** boundary.
- If $\frac{2\alpha}{c} > 1$ then $\mathcal{E} < \infty$ and ∞ is a **regular reflecting** boundary.

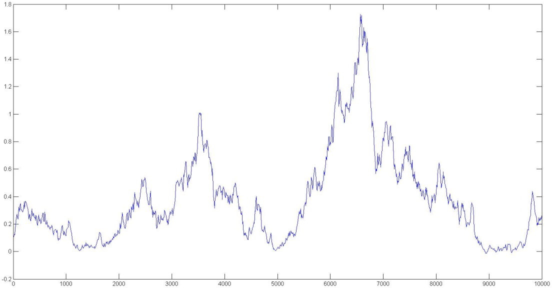


Figure: simulation of a Ψ-generalized Feller diffusion U reflected at 0, $\Psi(u) \sim -\alpha/\log(1/u)$ as u goes to 0.

Theorem (Stationarity)

Assume Ψ *of the form*

$$\Psi(z) = -\lambda - \delta z - \int_0^\infty (1 - e^{-zu})\pi(\mathrm{d}u)$$

with $\lambda \geq 0$, $\delta \geq 0$ *and* $\int_0^\infty (1 \wedge u)\pi(\mathrm{d}u) < \infty$. *Assume* $0 \leq \frac{2\lambda}{c} < 1$ *and define the condition*

$$\textbf{(A)} : (\delta = 0 \text{ and } \bar{\pi}(0) + \lambda \leq c/2).$$

- *If* **(A)** *holds then* $(Z_t, t \geq 0)$ *converges in probab. to* 0.
- *If* **(A)** *is not satisfied then* $(Z_t, t \geq 0)$ *converges in law towards the distribution carried over* $(\frac{2\delta}{c}, \infty)$ *whose Laplace transform is*

$$x \in \mathbb{R}_+ \mapsto \mathbb{E}[e^{-xZ_\infty}] := \frac{\int_x^\infty \exp\left(\int_\theta^y \frac{2\Psi(z)}{cz}\mathrm{d}z\right)\mathrm{d}y}{\int_0^\infty \exp\left(\int_\theta^y \frac{2\Psi(z)}{cz}\mathrm{d}z\right)\mathrm{d}y}.$$

Theorem (long-time behavior, $0 \leq \frac{2\lambda}{c} < 1$)

Let $(Z_t, t \geq 0)$ *be the extended process started from* $z \in (0, \infty)$.

1) *If* $0 \leq \frac{2\lambda}{c} < 1$ *and* $\Psi(z) \geq 0$ *for a certain* $z > 0$ *then*
   - 1-1) *If* $\int^{\infty} \frac{\mathrm{d}u}{\Psi(u)} = \infty$, *then* $Z_t > 0$ *for all* $t \geq 0$ *a.s. and* $Z_t \underset{t\to\infty}{\longrightarrow} 0$ *a.s.*
   - 1-2) *If* $\int^{\infty} \frac{\mathrm{d}u}{\Psi(u)} < \infty$, *then* $(Z_t, t \geq 0)$ *is absorbed at* 0 *in a finite time a.s..*

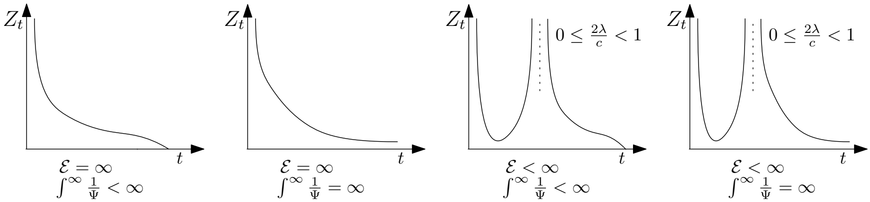


Figure: Schematic representation of the two behaviors at 0

Theorem (long-time behavior, $\frac{2\lambda}{c} \geq 1$)

2) *If* $\frac{2\lambda}{c} \geq 1$ *and* $\Psi(z) < 0$ *for all* $z > 0$ *then* $(Z_t, t \geq 0)$ *is absorbed at* ∞ *in finite time a.s.*

3) *If* $\frac{2\lambda}{c} \geq 1$ *and* $\Psi(z) \geq 0$ *for a certain* $z > 0$ *then*

$$\mathbb{P}_z(Z_t \underset{t\to\infty}{\longrightarrow} 0) = 1 - \mathbb{P}_z(\zeta_\infty < \infty)$$

$$= \frac{\int_0^\infty \frac{e^{-zu}}{u} \exp\left(-\int_\theta^u \frac{2\Psi(v)}{cv}\mathrm{d}v\right)\mathrm{d}u}{\int_0^\infty \frac{1}{u} \exp\left(-\int_\theta^u \frac{2\Psi(v)}{cv}\mathrm{d}v\right)\mathrm{d}u} \in (0,1).$$

and $Z_t > 0$ *for all* $t \geq 0$ *a.s. iff* $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} = \infty$.

Proofs: construction of the extensions

A first duality result

Recall $e_x(z) = e_z(x) = e^{-xz}$ and $\mathcal{L}e_x(z) = \mathcal{A}e_z(x)$ for $x, z \in (0, \infty)$.

Lemma (Laplace's duality lemma)

Assume $\mathcal{E} = \infty$. $\forall z \in [0, \infty)$, $\forall x \in (0, \infty)$,

$$\mathbb{E}_z[e^{-xZ_t^{min}}] = \mathbb{E}_x[e^{-zU_t}].$$

Sketch of proof.

See Ethier and Kurtz (corollary 4.15 p196). Assume U and $Z^{\min}$ independent, by the martingale problems for U and $Z^{\min}$, using that $Z^{\min}$ does not explode and that 0 is an exit for U, we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}\left(e^{-U_{t-s}Z_s^{\min}}\right) = \mathbb{E}\left(\mathcal{L}e_{U_{t-s}}(Z_s^{\min}) - \mathcal{A}e_{Z_s^{\min}}(U_{t-s})\right) = 0.$$

Hence

$$\mathbb{E}\left(e^{-U_{t-s}Z_s^{\min}}\right) = \mathbb{E}\left(e^{-U_t z}\right) = \mathbb{E}\left(e^{-xZ_t^{\min}}\right).$$

□

Theorem (Stationarity)

Assume Ψ *of the form*

$$\Psi(z) = -\lambda - \delta z - \int_0^{\infty}(1 - e^{-zu})\pi(\mathrm{d}u)$$

with $\lambda \geq 0$, $\delta \geq 0$ *and* $\int_0^{\infty}(1 \wedge u)\pi(\mathrm{d}u) < \infty$. *Assume* $0 \leq \frac{2\lambda}{c} < 1$ *and define the condition*

$$\textbf{(A)} : (\delta = 0 \text{ and } \bar{\pi}(0) + \lambda \leq c/2).$$

- *If* **(A)** *holds then* $(Z_t, t \geq 0)$ *converges in probab. to* 0.
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Infinity as an Entrance Boundary: sketch of proof

Recall

$$\mathbb{E}_z[e^{-xZ_t^{\min}}] = \mathbb{E}_x[e^{-zU_t}].$$

- Set $P_t e_x(z) := \mathbb{E}_z[e^{-xZ_t^{\min}}]$ for $z \in [0, \infty[$ and

$$P_t e_x(\infty) := \lim_{z\to\infty} \mathbb{E}_z[e^{-xZ_t^{\min}}] = \mathbb{P}_x(U_t = 0).$$

- Since $\mathcal{E} = \infty$, 0 is an exit of U and $\mathbb{P}_x(U_t = 0) > 0$.
- $x \mapsto \mathbb{P}_x(U_t = 0) = \mathbb{P}_x(\tau_0 \leq t)$ is the Laplace transform of a probability **entrance law** η_t, i.e. for any $t > 0$ and $s \geq 0$

$$\eta_{t+s} = \eta_t P_s$$

and $(P_t, t \geq 0)$ is a Feller semigroup on $[0, \infty]$.

- Let $(Z_t, t \geq 0)$ be a Feller process with semigroup $(P_t, t \geq 0)$. By definition: for any $z \in [0, \infty)$:

$$\mathbb{E}_z[e^{-xZ_t}] = \mathbb{E}_x[e^{-zU_t}]$$

and

$$\mathbb{E}_\infty[e^{-xZ_t}] = \mathbb{E}_x[e^{-\infty \cdot U_t}] = \mathbb{P}_x(U_t = 0).$$

Hence it has ∞ as entrance boundary.

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$$\begin{aligned}\mathbb{P}_\infty(Z_t < \infty) &= \lim_{x \to 0^+} \mathbb{E}_\infty(e^{-xZ_t}) \\ &= \lim_{x \to 0^+} \mathbb{P}_x(U_t = 0) \\ &= \mathbb{P}_{0^+}(U_t = 0) = 1.\end{aligned}$$

Hence ∞ is instantaneous.

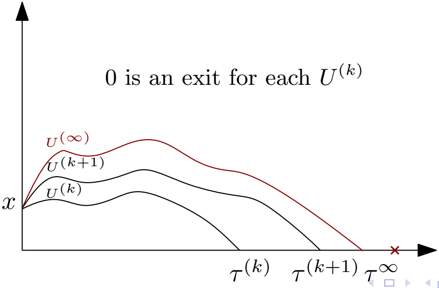
Infinity as regular reflecting boundary: sketch of proof.

Assume $\mathcal{E} < \infty$ and $\frac{2\lambda}{c} < 1$. Set $U_t^{(k)}$ the Ψ_k-generalized Feller diffusion and $Z^{(k)}$ the LCSBP(Ψ_k, c): $Z^{(k)}$ does not explode and by the previous result has ∞ entrance and

$$\mathbb{E}_z[e^{-xZ_t^{(k)}}] = \mathbb{E}_x[e^{-zU_t^{(k)}}]$$

where $U^{(k)}$ has 0 exit.

- For all x, $\Psi_{k+1}(x) \leq \Psi_k(x)$ so by the comparison theorem: $U_t^{(k+1)} \geq U_t^{(k)}$ for all t a.s. and $U_t^{(k)} \to U_t^{(\infty)}$ as $k \to \infty$.



- Since $||\mathcal{A}^{(k)}f - \mathcal{A}f||_\infty \to 0$ for any $f \in C_c^2$,

$$(U_t^{(\infty)}, t \leq \tau^\infty) \overset{law}{=} \text{ the minimal diffusion with generator } \mathcal{A}$$

$\tau^\infty := \inf\{t; U_t^{(\infty)} = 0\}$ and $\mathbb{P}_x(\tau^\infty < \infty) > 0$ since $\frac{2\lambda}{c} < 1$.

- Since $\tau^\infty \geq \tau^{(k)}$ and 0 is an exit of $(U_t^{(k)}, t \geq 0)$, on $\{\tau^\infty < \infty\}$

$$U^{(\infty)}_{t+\tau^\infty} = \lim U^{(k)}_{t+\tau^\infty} = 0.$$

Thus $(U_t^{(\infty)}, t \geq 0)$ has 0 regular absorbing.

To sum up, when $\mathcal{E} < \infty$ and $\frac{2\lambda}{c} < 1$

$$\mathbb{E}_x[e^{-zU_t^{(k)}}] \underset{k\to\infty}{\longrightarrow} \mathbb{E}_x[e^{-zU_t^{\mathrm{a}}}]$$

where $(U_t^{\mathrm{a}}, t \geq 0)$ is the Ψ-generalized Feller diffusion with 0 **regular absorbing**.

Let $(P_t^{(k)}, t \geq 0)$ the semi-group of $(Z_t^{(k)}, t \geq 0)$. Set

$$P_t e_x(z) := \lim_{k\to\infty} P_t^{(k)} e_x(z) = \mathbb{E}_x[e^{-zU_t^{\mathrm{a}}}].$$

Stone-Weierstrass entails that $P_t C_b \subset C_b$, where $C_b := C([0,\infty], \mathbb{R})$. One has

$$||P_t^{(k)} e_x - P_t e_x||_\infty = \sup_{z\in[0,\infty]} \left(\mathbb{E}_x[e^{-zU_t^{(k)}}] - \mathbb{E}_x[e^{-zU_t^{\mathrm{a}}}]\right) \underset{k\to\infty}{\longrightarrow} 0$$

Stone-Weierstrass again entails $||P_t^{(k)} f - P_t f||_\infty \longrightarrow 0$ for any $f \in C_b$ and

- $(P_t, t \geq 0)$ is a semigroup with the Feller property.
- Unif. conv. of semigroups implies convergence in $\mathbb{D}$ (Ethier-Kurtz (Thm 2.5 p167)), thus:

$$(Z_t^{(k)}, t \geq 0) \Longrightarrow (Z_t, t \geq 0)$$

Let $(Z_t, t \geq 0)$ be the Markov process on $[0,\infty]$ with semigroup $(P_t, t \geq 0)$. One has

$$\mathbb{E}_z[e^{-xZ_t}] = \mathbb{E}_x[e^{-zU_t^{\mathrm{a}}}].$$

It remains to show that Z is an extension of $Z^{\min}$. One has for any $f \in C_c^2$,

$$||\mathcal{L}^{(k)} f - \mathcal{L} f||_\infty \to 0 \text{ as } k \to \infty$$

and thus $(Z_{t\wedge\zeta_\infty}, t \geq 0)$ solves **(MP)**. By well-posedness,

$$(Z_{t\wedge\zeta_\infty}, t \geq 0) \overset{law}{=} (Z_t^{\min}, t \geq 0).$$

Conclusion: when $\mathcal{E} < \infty$ and $\frac{2\lambda}{c} < 1$, ∞ is accessible and

$$\mathbb{E}_\infty[e^{-xZ_t}] = \mathbb{P}_x(U_t^0 = 0) = \mathbb{P}_x(\tau_0 \leq t) > 0$$

and ∞ is regular for Z. Moreover for any $z \in [0, \infty]$,

$$\mathbb{P}_z(Z_t < \infty) = \mathbb{E}_{0+}[e^{-zU_t^0}] = 1$$

and ∞ is reflecting.

Facts

- What happens in the process past explosion is entirely encoded in the law of the first hitting time of 0 of U.
- We have so far not obtained precise information on the first explosion time, the local time at ∞ and the excursion measure. The construction given previously of the LCSBP Z reflected at ∞ does not allow us to describe the latters.

In the remaining time, we now state some results in this direction.

In order to go further in the description of the reflected process Z, we will use a second duality relationship: for any $x, y \in (0, \infty)$ and $t \geq 0$,

$$\mathbb{P}_x(U_t < y) = \mathbb{P}_y(x < V_t), \tag{3}$$

where the process $(V_t, t \geq 0)$ is the so-called Siegmund dual diffusion of U.

$$Z \overset{\textbf{Laplace dual}}{\longleftrightarrow} U \overset{\textbf{Siegmund dual}}{\longleftrightarrow} V. \tag{4}$$

By combining the two dualities one can check that for any $t \geq 0$ and all $z, x \in (0, \infty)$,

$$\mathbb{E}_z(e^{-xZ_t}) = \int_0^\infty z e^{-zy} \mathbb{P}_y(V_t > x) \mathrm{d}y. \tag{5}$$

Digression: Siegmund duality for one-dimensional diffusions

Theorem (Diffusions and Siegmund duality)

Let σ^2 be a C^1 strictly positive function on $(0,\infty)$ and μ be a continuous function on $(0,\infty)$. Let $(U_t, t \geq 0)$ be a diffusion over $(0,\infty)$ with generator

$$\mathscr{A} f(x) := \tfrac{1}{2}\sigma^2(x) f''(x) + \mu(x) f'(x)$$

such that ∞ is either inaccessible (entrance or natural) or absorbing (exit or regular absorbing). Then for any $0 < u, v < \infty$ and any $t \geq 0$

$$\mathbb{P}_u(U_t < v) = \mathbb{P}_v(V_t > u), \tag{6}$$

with $(V_t, t \geq 0)$ the diffusion whose generator is

$$\mathscr{G} f(x) := \frac{1}{2}\sigma^2(x) f''(x) + \left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\sigma^2(x) - \mu(x)\right) f'(x). \tag{7}$$

Let S_U and M_U be the scale function and the speed measure of U. Up to some irrelevant multiplicative constants, we have the equalities

$$S_U = M_V, M_U = S_V.$$

The following correspondences for boundaries and longterm behaviors of U and V hold:

U	V
0 exit	0 entrance
0 regular absorbing	0 regular reflecting
0 entrance	0 exit
∞ exit	∞ entrance
∞ & 0 attracting	positive recurrence

Table: Boundaries of U, V.

Proposition

The Siegmund dual of $(U_t, t \geq 0)$ is the diffusion $(V_t, t \geq 0)$ solution to an SDE of the form

$$\mathrm{d}V_t = \sqrt{cV_t}\mathrm{d}B_t + (c/2 + \Psi(V_t))\mathrm{d}t, \ \ V_0 = y \in (0, \infty), \qquad (8)$$

where $(B_t, t \geq 0)$ is some Brownian motion and whose boundary condition at 0 and ∞ are given in correspondence with that of U in the following way:

Integral condition	*Boundary of U*	*Boundary of V*
$\mathcal{E} = \infty$	0 *exit*	0 *entrance*
$\mathcal{E} < \infty$ & $2\lambda/c < 1$	0 *regular absorbing*	0 *regular reflecting*
$2\lambda/c \geq 1$	0 *entrance*	0 *exit*
$\int^{\infty} \frac{\mathrm{d}x}{\Psi(x)} = \infty$	∞ *natural*	∞ *natural*
$\int^{\infty} \frac{\mathrm{d}x}{\Psi(x)} < \infty$	∞ *entrance*	∞ *exit*

Gathering the correspondences displayed in previous tables, we obtain the following ones between V and Z. Notice that the boundaries 0 and ∞ are exchanged but the behaviors of the processes are not anymore.

Boundary of V	Boundary of Z
0 entrance	∞ entrance
0 regular reflecting	∞ regular reflecting
0 exit	∞ exit
∞ natural	0 natural
∞ exit	0 exit

Table: Boundaries of V, Z.

Denote by T_y the first hitting time of $y \in [0, \infty]$ of the diffusion $(V_t, t \geq 0)$ and set $\mathscr{G}$ its generator:

$$\mathscr{G} f(x) := \frac{c}{2} x f''(x) + \left(\frac{c}{2} + \Psi(x)\right) f'(x). \tag{9}$$

Then, from the general theory of one-dimensional diffusions, the Laplace transform of T_y is expressed, for any $\theta > 0$, as

$$\mathbb{E}_x[e^{-\theta T_y}] = \begin{cases} \frac{h_\theta^+(x)}{h_\theta^+(y)}, & x \leq y \\ \frac{h_\theta^-(x)}{h_\theta^-(y)}, & x \geq y, \end{cases} \tag{10}$$

and functions h_θ^- and h_θ^+ are C^2 and respectively decreasing and increasing solutions to the equation

$$\mathscr{G} h(x) := \frac{c}{2} x h''(x) + \left(\frac{c}{2} + \Psi(x)\right) h'(x) = \theta h(x), \text{ for all } x \in (0, \infty). \tag{11}$$

with appropriate boundary conditions at non natural boundary points.

Denote by $\mathbb{e}_z$ an exponential random variable independent of V with parameter z, and by $T_y^{\mathbb{e}_z}$ the first hitting time of point y by the diffusion V started from $\mathbb{e}_z$.

Theorem (Laplace transform of the extinction time of LCSBPs)

For any $0 < z < \infty$ *and* $\theta > 0$,

$$\mathbb{E}_z[e^{-\theta\zeta_0}] = \int_0^\infty ze^{-zx}\frac{h_\theta^+(x)}{h_\theta^+(\infty)}\mathrm{d}x = \mathbb{E}[e^{-\theta T_\infty^{\mathbb{e}_z}}] \in [0,\infty) \tag{12}$$

In particular, if ∞ *is not absorbing for* Z *(i.e. if* $2\lambda/c < 1$*) then*

$$\mathbb{E}_\infty[e^{-\theta\zeta_0}] = \mathbb{E}_0[e^{-\theta T_\infty}] \in (0,\infty).$$

Theorem (Laplace transform of the first explosion time of LCSBPs)

$$\mathbb{E}_z[e^{-\theta\zeta_\infty}] = \int_0^\infty z e^{-zx} \frac{h_\theta^-(x)}{h_\theta^-(0)} \mathrm{d}x = \mathbb{E}[e^{-\theta T_0^{\mathbb{e}_z}}] \in [0,\infty).$$

Theorem (Local time)

Assume ∞ regular reflecting ($\mathcal{E} < \infty$ & $2\lambda/c < 1$),

1. *the local time at ∞ of Z, $(L_t^Z, t \geq 0)$ has the same law as the local time of V at 0, $(L_t^V, t \geq 0)$, for a certain deterministic multiplicative factor.*
2. *the Laplace exponent of the inverse local time subordinator $(\tau_x^Z, 0 \leq x < \xi)$ is $\kappa_Z : \theta \mapsto 1/h_\theta^-(0)$.*
3. *In addition,*

$$\kappa_Z(0) = 1/S_Z(0)$$

with $S_Z(0) := \int_0^\infty \frac{1}{c}\frac{\mathrm{d}x}{x} e^{-\int_{x_0}^x \frac{2\Psi(y)}{cy}\mathrm{d}y} \in (0,\infty]$,

Corollary

Assume $\mathcal{E} < \infty$ & $\frac{2\lambda}{c} < 1$, *set* $\mathcal{I} := \{t > 0 : Z_t = \infty\}$,

$$\dim_H(\mathcal{I}) = 2\lambda/c \in [0, 1) \ a.s.$$

Example

1. A specific example is given by the case $\Psi \equiv -\lambda$ with $\lambda > 0$. In this setting, the diffusion V is solution to the SDE

$$\mathrm{d}V_t = \sqrt{cV_t}\mathrm{d}B_t + (c/2 - \lambda)\mathrm{d}t.$$

   Therefore, V is a squared Bessel diffusion with non-negative dimension and the inverse local time at 0 of V is a stable subordinator with index $2\lambda/c$

2. If $\Psi(x) \underset{x \to 0+}{\sim} -\alpha/\log(1/x)$. One has $\mathcal{E} < \infty$ and by the corollary , $\dim_H(\mathcal{I}) = 0$ a.s..

Theorem (Excursion measure)

Assume ∞ regular reflecting ($\mathcal{E} < \infty$ & $2\lambda/c < 1$). For any $x \in [0, \infty)$ and $q > 0$,

$$n_Z\left(\int_0^{\zeta} e^{-qu} e^{-x\epsilon(u)} \mathrm{d}u\right) = n_V\left(\int_0^{\ell} e^{-qu} \mathbb{1}_{(x,\infty)}(\omega(u)) \mathrm{d}u\right). \quad (13)$$

Moreover,

$$n_Z\left(\int_0^{\zeta} e^{-x\epsilon(u)} \mathrm{d}u\right) = \int_x^{\infty} e^{\int_{x_0}^{y} \frac{2\Psi(u)}{cu} \mathrm{d}u} \mathrm{d}y \in (0, \infty]. \quad (14)$$

Theorem

Assume ∞ regular reflecting ($\mathcal{E} < \infty$ & $2\lambda/c < 1$) and that $-\Psi$ is not the Laplace exponent of a subordinator. Denote by I the infimum of an excursion of Z. Its law under n_Z is given by

$$n_Z(I \leq a) = 1/S_Z(a),$$

with $S_Z(a) := \int_0^{\infty} \frac{1}{c}\frac{\mathrm{d}x}{x} e^{-ax} e^{-\int_{x_0}^{x} \frac{2\Psi(u)}{cu}\mathrm{d}u}$ *for all* $a \geq 0$.

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Conclusion of the short course

We have seen two different uses of stochastic duality with respect to a function:

$$\mathbb{E}_x[H(X_t, y)] = \mathbb{E}_y[H(x, Y_t)]$$

- In the first two talks, we have investigated the flow of CSBPs X and its Siegmund dual $Y := \hat{X}$, the dual function was
$$H(x, y) = \mathbb{1}_{\{x \leq y\}}.$$
The duality was made pathwise by the use of Bertoin-Le Gall's flow; and related to the genealogy backwards in time.
- In the last talk; we have used first a duality relationship with $X = Z$, the LCSBP, $Y = U$ the Ψ-generalized Feller diffusion with function
$$H(z, x) = e^{-xz}$$
for studying the LCSBP and its recurrent extensions of the process past explosion as well as its long-term behavior. We then use a second dual process V, Siegmund dual of U, for studying deeper the process.

Thank you for your attention