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On the hitting times of continuous-state branching processes with immigration

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Abstract

We study a two-dimensional joint distribution related to the first passage time below a level for a continuous-state branching process with immigration. We provide an explicit expression of its Laplace transform and obtain a necessary and sufficient criterion for transience or recurrence. We follow the approach of Shiga (1990), by finding some λ-invariant functions for the generator.

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1. Introduction

The continuous-state branching processes with immigration (CBI for short) are a class of time-homogeneous Markov processes with values in $\mathbb{R}_+$. They have been introduced by Kawazu

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and Watanabe in 1971, see [14], as limits of rescaled Galton–Watson processes with immigration. Continuous-state branching processes with immigration form a class of Markov processes which has received significant attention in the literature. Any CBI process is characterized in law by the Laplace exponents Ψ and Φ of two independent Lévy processes: a spectrally positive process (which describes the reproduction) and a subordinator (which describes the immigration). The purpose of this paper is to give and exploit an explicit formula for the Laplace transform of the first passage time below a level. In particular we provide a necessary and sufficient condition for a CBI process to be recurrent or transient.

We adopt the following definition of recurrence and transience.

Definition. We say that the process $(X_t, t \geq 0)$ is recurrent if there exists $x \in \mathbb{R}_+$ such that

$$\mathbb{P}_x(\liminf_{t\to\infty} |X_t - x| = 0) = 1. \tag{1}$$

On the other hand, we say that the process is transient if

$$\mathbb{P}_x(\lim_{t\to\infty} X_t = \infty) = 1 \text{ for every } x \in \mathbb{R}_+. \tag{2}$$

When the mechanisms of reproduction and immigration reduce to $\Psi(q) = \frac{\sigma^2}{2}q^2$ and $\Phi(q) = bq$, the process is the Feller diffusion, also called the Cox–Ingersoll–Ross model in the financial setting. This is the unique solution to the stochastic equation:

$$X_t = x + \sigma \int_0^t \sqrt{X_s} \mathrm{d}B_s + bt,$$

where $(B_t, t \geq 0)$ is a Brownian motion. A standard method to study the hitting times, as well as the transience and recurrence of a general diffusion, is to use potential theory and scale functions (see for instance pages 128–129 of Itō and Mckean [13]). This theory yields the following classic result concerning the Feller diffusion $(X_t, t \geq 0)$: if $2b \geq \sigma^2$, then the point 0 is polar. If $2b > \sigma^2$, the process is transient, otherwise the process is recurrent. In particular, if $2b = \sigma^2$, then 0 is polar and the process is recurrent (we refer, for instance, to Chapter XI of Revuz–Yor [27] for a proof).

We shall study these path-properties for the general CBI processes. The polarity of zero has been studied in Foucart and Uribe Bravo [11]. However, this latter work focuses on the zero-set and does not provide a criterion for transience or recurrence of the process. Moreover, as we shall see, zero may be polar and recurrent (in the sense of (1)).

On the one hand, when the mechanism Ψ reduces to $\Psi(q) = \gamma q$ with $\gamma > 0$, the class of CBI processes corresponds to positive Ornstein–Uhlenbeck processes. This class of processes has been intensively studied. Hadjiev [12] get a formula for the hitting times of generalized Ornstein–Uhlenbeck processes. Novikov [22], Patie [23,24] apply potential theory to get some identities for the joint law of $(\sigma_a, \int_0^{\sigma_a} X_s \mathrm{d}s)$, and for the first exit times. On the other hand, when no immigration is taken into account (namely, with $\Phi \equiv 0$), the corresponding CBI process is simply a continuous-state branching process (CB process) for which many results have been obtained by using the Lamperti transform (which relates any CB process to a spectrally positive Lévy process). We refer, for instance, to Chapter 10 of [16]. We mention that a Lamperti-type representation for the CBI processes has been obtained by Caballero et al. in [6]. However, our methods do not rely on this representation.

Our aim is to generalize some of these results when immigration is taken into account for a general reproduction mechanism Ψ. In this framework, the integral from 0 to σ_a of the process can be interpreted as the total population up to time σ_a. The results reveal the interplay between Φ and Ψ in some path properties of CBI processes.

The paper is organized as follows. In Section 2, we recall the definition as well as some known properties of continuous-state branching processes. Section 3 discusses our main results. We study the state space of a CBI in Section 4. In Section 5 we prove a key lemma providing some λ-invariant functions, and apply it to establish our first theorem. In Section 6, we show a formula for the Laplace transform of the hitting times and study the polarity of 0. In Section 7, we establish our criterion for recurrence or transience and we study the law of the infimum of a transient CBI. We end the section by showing how to construct null-recurrent CBIs. In Section 8, we study the integral of the CBI process up to time σ_a.

Throughout the article, we take the convention that for any finite real number C, $C/\infty = 0$.

2. Preliminaries

For an introduction to continuous-state branching processes with immigration, we refer to Li [20,21] and Kyprianou [16]. We recall here some of their fundamental properties.

2.1. Continuous-state branching processes

Let $x \in \mathbb{R}_+$. Notation $\mathbb{P}_x$ denotes the law of the process started at $x \in \mathbb{R}_+$, and $\mathbb{E}_x$ the corresponding expectation operator. A continuous-state branching process $(Z_t, t \geq 0)$ is a Markov process which satisfies the following property: for any x and $y \in \mathbb{R}_+$, the process $(Z_t, t \geq 0)$ starting from $x + y$ has the same law as

$$(Z_t^1 + Z_t^2, t \geq 0),$$

where $(Z_t^1, t \geq 0)$ and $(Z_t^2, t \geq 0)$ are two independent copies of $(Z_t, t \geq 0)$ starting respectively at x and y. There exists a unique function Ψ of the form:

$$\Psi(q) = \gamma q + \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu1_{\{u\in(0,1)\}})\pi(\mathrm{d}u)$$

such that the Laplace transform of the one-dimensional distribution of $(Z_t, t \geq 0)$ is given by

$$\mathbb{E}_x[e^{-\lambda Z_t}] = \exp(-x v_t(\lambda))$$

where the map $t \mapsto v_t(\lambda)$ is the solution to the differential equation

$$\frac{\partial}{\partial t} v_t(\lambda) = -\Psi(v_t(\lambda)), \qquad v_0(\lambda) = \lambda.$$

Recall the following classification (see Chapter 10 of [16] for details): the branching mechanism Ψ is said to be subcritical if $\Psi'(0+) > 0$, critical if $\Psi'(0+) = 0$ and supercritical if $\Psi'(0+) < 0$. For any $x \geq 0$, define the extinction time of $(Z_t, t \geq 0)$ by

$$\zeta := \inf\{t \geq 0; Z_t = 0\} \in [0, \infty].$$

The extinction time has the following law (see for instance Theorem 3.5 and 3.8 pages 59–60 of [21]):

$$\mathbb{P}_x[\zeta \leq t] = \exp(-x v_t)$$

with $v_t = \lim_{\lambda\to\infty} v_t(\lambda)$.

The asymptotic behavior of a CB(Ψ) process is well understood. Indeed, the CB(Ψ) process $(Z_t, t \geq 0)$ is absorbed in 0 with positive probability if and only if there exists $\theta > 0$ such that $\Psi(z) > 0$ for $z \geq \theta$ and

$$\int_\theta^\infty \frac{\mathrm{d}q}{\Psi(q)} < \infty. \tag{3}$$

Under this condition, $v_t < \infty$ for all t and $v := \lim_{t\to\infty} v_t \in [0, \infty[$ is the largest root of the equation $\Psi(q) = 0$ and

$$\mathbb{P}_x[\zeta < \infty] = \exp(-xv).$$

In the (sub)critical case $v = 0$ and the process is absorbed in zero almost surely. In the supercritical case, we have $v > 0$ and the process goes to infinity with positive probability. Lastly, the process is absorbed in ∞ in finite time if and only if

$$\int_0^1 \frac{\mathrm{d}q}{|\Psi(q)|} < \infty. \tag{4}$$

2.2. Continuous-state branching processes with immigration

A continuous-state branching process with immigration is a Markov process $(X_t, t \geq 0)$ whose law is characterized by two functions of the variable $q \geq 0$:

$$\Psi(q) = \gamma q + \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu1_{\{u\in(0,1)\}})\pi(\mathrm{d}u), \tag{5}$$

$$\Phi(q) = bq + \int_0^\infty (1 - e^{-qu})\nu(\mathrm{d}u), \tag{6}$$

where $\sigma, b \geq 0$, $\gamma \in \mathbb{R}$ and ν, π are two Lévy measures such that $\int_0^\infty (1 \wedge u)\nu(\mathrm{d}u) < \infty$ and $\int_0^\infty (1 \wedge u^2)\pi(\mathrm{d}u) < \infty$. The measure ν characterizes the jumps of the subordinator that describes the arrival of immigrants in the population. The non-negative constants σ and b correspond respectively to the continuous reproduction and the continuous immigration. To simplify our notation, a continuous-state branching process with reproduction mechanism Ψ and immigration mechanism Φ is called the CBI(Ψ, Φ) process.

Let $(X_t, t \geq 0)$ be a CBI(Ψ, Φ) process, its one-dimensional marginal law satisfies:

$$\mathbb{E}_x[e^{-qX_t}] = \exp\left(-xv_t(q) - \int_0^t \Phi(v_s(q))\mathrm{d}s\right), \tag{7}$$

with $\frac{\partial v_t(q)}{\partial t} = -\Psi(v_t(q))$ and $v_0(q) = q$.
A CBI(Ψ, Φ) process $(X_t, t \geq 0)$ starting from $x + y$ has the same law as

$$(X_t^1 + X_t^2, t \geq 0),$$

where $(X_t^1, t \geq 0)$ is a CBI(Ψ, Φ_1) process starting at x and $(X_t^2, t \geq 0)$ is an independent CBI(Ψ, Φ_2) process starting at y, with $\Phi_1 + \Phi_2 = \Phi$.

Denote by $C_0^2(\mathbb{R}_+)$ the space of twice-differentiable functions that tend to zero as x goes to ∞. Kawazu and Watanabe [14] establish that a CBI(Ψ, Φ) process is a Feller process with generator the operator L acting on $C_0^2(\mathbb{R}_+)$ as follows

$$Lf(x) := \frac{\sigma^2}{2} x f''(x) + (b - \gamma x) f'(x) + x \int_0^\infty (f(x+z) - f(x) - z 1_{[0,1]}(z) f'(x)) \pi(\mathrm{d}z) + \int_0^\infty (f(x+z) - f(x))\, \nu(\mathrm{d}z). \tag{8}$$

Apart if explicitly mentioned, we always assume that the CBI process is non-deterministic. This is equivalent to assume that one of the three conditions holds: $\sigma \neq 0$, $\pi \not\equiv 0$, or $\nu \not\equiv 0$. Moreover, we assume that there exists $q \in \mathbb{R}_+$ such that $\Psi(q) > 0$ (i.e. $-\Psi$ is not the Laplace exponent of a subordinator). This is equivalent to assume that the *effective drift* $\mathbf{d}$ defined by

$$\mathbf{d} := \begin{cases} \gamma + \int_0^1 z \pi(\mathrm{d}z) \text{ if the process has bounded variation paths} \\ \infty \text{ if the process has unbounded variation paths,} \end{cases} \tag{9}$$

belongs to $(0, \infty]$. Otherwise, the corresponding CBI process would be non-decreasing, and the problems studied in the present work are trivial.
We are now ready to state our main results.

3. Main results

Denote the first entrance time in $[0, a]$ by σ_a:

$$\sigma_a := \inf\{t > 0; X_t \le a\}. \tag{10}$$

Remark that the process has no downward jumps, therefore $X_{\sigma_a} = a$ almost surely. We will discuss the law of σ_a when the process starts from a state x greater than a. The first theorem is the following. Set $\ell := \frac{b}{\mathbf{d}}$ with $\mathbf{d}$ defined by (9).

Theorem 1. *Let $x > a \ge \ell$. For every $\lambda > 0$ and $\mu \ge 0$, we have*

$$\mathbb{E}_x\left[\exp\left\{-\lambda\sigma_a - \mu \int_0^{\sigma_a} X_t \mathrm{d}t\right\}\right] = \frac{\int_{q(\mu)}^\infty \frac{\mathrm{d}z}{\Psi(z)-\mu} \exp\left(-xz + \int_\theta^z \frac{\Phi(u)+\lambda}{\Psi(u)-\mu}\mathrm{d}u\right)}{\int_{q(\mu)}^\infty \frac{\mathrm{d}z}{\Psi(z)-\mu} \exp\left(-az + \int_\theta^z \frac{\Phi(u)+\lambda}{\Psi(u)-\mu}\mathrm{d}u\right)}, \tag{11}$$

where $q(\mu) := \sup\{q \ge 0 : \Psi(q) = \mu\}$, and θ is an arbitrary constant larger than $q(\mu)$.

When Φ is null or taken of the specific form Ψ', some formulas are simplified and we recover certain results on continuous-state branching processes.
We recover and complete some results of [11] through more classic techniques.

Theorem 2 (*Foucart and Uribe–Bravo [11]*). *The only point that may be polar is ℓ. If $\mathbf{d} < \infty$ then ℓ is polar. In the unbounded variation case, $\ell = 0$ and 0 is polar or hit with positive probability accordingly as*

$$\int_\theta^\infty \frac{\mathrm{d}z}{\Psi(z)} \exp\left[\int_\theta^z \frac{\Phi(x)}{\Psi(x)}\mathrm{d}x\right] = \infty \text{ or } < \infty. \tag{12}$$

The third theorem yields a necessary and sufficient criterion for the recurrence or transience property of a CBI(Ψ, Φ)process when $\Phi \not\equiv 0$. We mention that results in the same vein have been obtained by Chen et al. [8] in the setting of discrete branching Markov chains with immigration.

Theorem 3. (a) *In the critical or subcritical case, the process is recurrent or transient accordingly as*

$$\int_0^1 \frac{\mathrm{d}z}{\Psi(z)} \exp\left[-\int_z^1 \frac{\Phi(x)}{\Psi(x)}\mathrm{d}x\right] = \infty \text{ or } < \infty. \tag{13}$$

(b) *In the supercritical case, the* CBI(Ψ, Φ) *process is transient.*

For a critical CBI process with finite variance, we have a simpler criterion for its recurrence and transience, which somehow is reminiscent of the Feller diffusion.

Corollary 4. *Assume* $\int_1^\infty u^2 \log u \, \pi(\mathrm{d}u) < \infty$ *and* $\int_1^\infty u^2 \nu(\mathrm{d}u) < \infty$ *and consider*

$$\Psi(q) = \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu)\pi(\mathrm{d}u),$$
$$\Phi(q) = bq + \int_0^\infty (1 - e^{-qu})\nu(\mathrm{d}u).$$

Let $\tilde{\sigma}^2 := \Psi''(0)$ *and* $\tilde{b} := \Phi'(0)$. *The process is transient if and only if* $2\tilde{b} > \tilde{\sigma}^2$.

Remark 1. • The integrability condition $\int_\theta^\infty \frac{1}{\Psi(z)}\mathrm{d}z < \infty$ implies that $\mathbf{d} = \infty$, which entails that $\ell = 0$. However, it is worth mentioning that none of these implications are equivalences.

- In the criterion of Theorem 3, when the mechanism Ψ is subcritical, one can replace Φ by the map $q \mapsto \int_1^\infty (1 - e^{-qx})\nu(\mathrm{d}x)$ in (13). In other words, neither the continuous immigration nor its small jumps play a role for the process to be transient. Moreover, we should mention that when $\Psi(q) = \gamma q$, the criterion coincides with that of Shiga [29]. Note that a subcritical CBI with $\Phi(q) = bq$ is always recurrent.
- If the state 0 is not polar, then one has the same necessary and sufficient conditions for both neighborhood-recurrence and point-recurrence (studied in [11]) of the state 0.

Example 1. Consider $\Psi(q) = dq^\alpha$, $\Phi(q) = d'q^\beta$ with $\alpha \in (1, 2]$ and $\beta \in (0, 1]$.

- If $\beta > \alpha - 1$, the process is recurrent and 0 is polar.
- If $\beta < \alpha - 1$, the process is transient and 0 is not polar.
- If $\beta = \alpha - 1$ and $\alpha \in (1, 2]$, the process is recurrent if $d'/d \le \alpha - 1$ and transient if $d'/d > \alpha - 1$. The point 0 is polar if and only if $d'/d \ge \alpha - 1$. We observe that if $d'/d = \alpha - 1$, then 0 is polar but $\liminf_{t\to\infty} X_t = 0$. We point out that in this case, the CBI process is selfsimilar. Patie in [25] obtained the condition for 0 to be polar via other arguments.

4. State space of CBI processes

We study here the state space of a general CBI process. A trivial example of the CBI process which is not irreducible in $\mathbb{R}_+$ is the deterministic one. Namely, if $\Phi(q) = bq$ and $\Psi(q) = \gamma q$ with $\gamma > 0$, the associated CBI is $X_t = X_0 e^{-\gamma t} + \frac{b}{\gamma}(1 - e^{-\gamma t})$. The path of this process is above $\frac{b}{\gamma}$ as soon as $X_0 > \frac{b}{\gamma}$. As already mentioned the case when $-\Psi$ is the Laplace exponent of a subordinator is excluded. Recall $\mathbf{d} > 0$ and $\ell = \frac{b}{\mathbf{d}}$. We state a lower bound for any CBI process.

Proposition 5. *Let X be a* CBI(Ψ, Φ) *process started at $x \in (0, \infty)$. Then, $\mathbb{P}_x$ almost surely, for all $t > 0$,*

$$X_t \geq e^{-\mathbf{d}t} x + \ell \left(1 - e^{-\mathbf{d}t}\right). \tag{14}$$

In particular, this implies $\liminf_{t\to\infty} X_t \geq \ell$.

Proof. Firstly, one can notice that when X has unbounded variation paths, then $\mathbf{d} = \infty$ and $\ell = 0$. The lower bound in the lemma is then null and the statement is clear. We then focus on the case of bounded variation and denote, for all $t > 0$, $x_t := e^{-\mathbf{d}t} x + \ell \left(1 - e^{-\mathbf{d}t}\right)$. Using the càdlàg regularity, it will be sufficient to prove that for a fixed $t \in (0, \infty)$,

$$\mathbb{P}_x \left(X_t < x_t\right) = 0. \tag{15}$$

Let $(\tilde{X}_t, t \geq 0)$ be a CBI$(\Psi, \tilde{\Phi})$, where $\tilde{\Phi}(\lambda) = b\lambda$. We have then for all λ,

$$\mathbb{E}_x[e^{-\lambda \tilde{X}_t}] = \exp\left(-x v_t(\lambda) - b \int_0^t v_s(\lambda)\mathrm{d}s\right),$$

thus $\mathbb{E}_x[e^{-\lambda \tilde{X}_t}] \geq \mathbb{E}_x[e^{-\lambda X_t}]$, and therefore $\mathbb{P}_x(X_t < x_t) \leq \mathbb{P}_x(\tilde{X}_t < x_t)$. We will show that the latter probability is 0.

It is well-known that for a fixed $t > 0$, the map $\lambda \mapsto v_t(\lambda)$ is the Laplace exponent of a subordinator (see for instance Bertoin–Le Gall [2]). More precisely the underlying subordinator has for drift $e^{-\mathbf{d}t}$ (see Duquesne and Labbé in [10] Section 2.1 for details). Consider the Laplace exponent of the driftless subordinator:

$$w_t(\lambda) := v_t(\lambda) - e^{-\mathbf{d}t}\lambda.$$

One can write

$$\mathbb{E}_x[e^{-\lambda \tilde{X}_t}] = \exp\left(-\lambda x_t - x w_t(\lambda) - b \int_0^t w_s(\lambda)\mathrm{d}s\right), \tag{16}$$

and

$$\mathbb{E}_x[\exp -\lambda(\tilde{X}_t - x_t)] = \exp\left(-x w_t(\lambda) - b \int_0^t w_s(\lambda)\mathrm{d}s\right). \tag{17}$$

One can plainly check that the map $\lambda \mapsto x w_t(\lambda) + b \int_0^t w_s(\lambda)\mathrm{d}s$ is the Laplace exponent of a non-negative random variable. We deduce $\tilde{X}_t \geq x_t$, $\mathbb{P}_x$-a.s., and thus (15). □

Remark 2. Alternatively, one can use stochastic calculus. Consider the case of bounded variation paths, for which $\sigma = 0$ and $\int_0^1 x\pi(\mathrm{d}x) < \infty$. Let $N_0(\mathrm{d}s, \mathrm{d}u)$ and $N_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ be two independent Poisson random measures on $(0, \infty)^2$ and $(0, \infty)^3$ with intensity $\mathrm{d}s\nu(\mathrm{d}z)$ and $\mathrm{d}s\pi(\mathrm{d}z)\mathrm{d}u$, respectively. For each $x \geq 0$ there is a pathwise unique positive strong solution to the following stochastic equation:

$$X_t = x + \int_0^t (b - \mathbf{d}X_s)\mathrm{d}s + \int_0^t \int_0^\infty z N_0(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_0^\infty \int_0^{X_{s-}} z N_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).$$

By Itô's formula, the solution $(X_t, t \geq 0)$ is a CBI (Ψ, Φ) with $\sigma = 0$; see Theorem 3.1 of Dawson and Li [9]. On the other hand, let $(x_t, t \geq 0)$ be the solution to the ordinary differential equation:

$$x_t = x + \int_0^t (b - \mathbf{d} x_s)\mathrm{d}s.$$

It follows from Theorem 2.2 of Dawson and Li [9] that $\mathbb{P}_x(X_t \geq x_t \text{ for all } t \geq 0) = 1$.

In the (sub)critical case, a necessary and sufficient condition for the existence of a stationary distribution was announced by Pinsky [26] and obtained by Li:

Theorem 6 (*Theorem 3.20 in Li [21]*)**.** (i) *If* $\int_0^1 \frac{\Phi(u)}{\Psi(u)}\mathrm{d}u < \infty$, *then the* CBI$(\Psi, \Phi)$ *process,* $(X_t, t \geq 0)$*, has an invariant probability distribution. In the subcritical case* ($\Psi'(0+) > 0$)*, this integral condition is equivalent to*

$$\int_1^\infty \log(u)\nu(\mathrm{d}u) < \infty.$$

(ii) *If* $\int_0^1 \frac{\Phi(u)}{\Psi(u)}\mathrm{d}u = \infty$, *then for all* $x, b \in \mathbb{R}_+$,

$$\lim_{t\to\infty} \mathbb{P}_x(X_t \leq b) = 0.$$

Remark 3.
- Following the usual terminology, a CBI process with a non-degenerate limit distribution is said to be positive recurrent. One can easily check that any positive recurrent CBI process is indeed recurrent in the sense of (1) by using Theorem 3.
- The second statement of Theorem 6 is not plainly stated in [21]. Nevertheless, one can observe in the proof of Theorem 3.20 in [21] that if $\int_0^1 \frac{\Phi(u)}{\Psi(u)}\mathrm{d}u = \infty$, then $\mathbb{E}_x\left[e^{-\lambda X_t}\right] \underset{t\to\infty}{\longrightarrow} 0$. We refer also to Appendix A of Keller-Ressel and Mijatović [15].

It follows from Theorem 6 and Proposition 4.4 of [15] that either $(X_t, t \geq 0)$ has the non-degenerate limit distribution with support $[\ell, \infty)$ or $X_t \to \infty$ in probability as $t \to \infty$. Thus, applying Fatou's lemma, it is not hard to see that

$$\mathbb{P}_x\left(\limsup_{t\to\infty} X_t = \infty\right) = 1, \quad \text{for any } x \in \mathbb{R}_+, \tag{18}$$

when $\Phi \not\equiv 0$. Starting from a point in $\mathcal{S} = [\ell, \infty)$, the process stays in $\mathcal{S}$, so we shall work with $\mathcal{S}$ as the state space.

5. Proof of Theorem 1

Recall that L is the infinitesimal generator of a CBI(Ψ, Φ) stated in (8). Let $\mu \geq 0$ and set $\bar{\Psi}(q) = \Psi(q) - \mu$. Denote $\bar{L}$ the generator of $(\bar{X}_t, t \geq 0)$, a CBI$(\bar{\Psi}, \Phi)$. For all $f \in C_0^2(\mathbb{R}_+)$

$$\bar{L}f(x) = Lf(x) - \mu x f(x).$$

Recall $q(\mu) = \sup\{q \geq 0 : \Psi(q) = \mu\}$. Note that $q(\mu) < \infty$ since by assumption there exists q such that $\Psi(q) > 0$. We fix a constant $\theta = \theta(\mu) \in (q(\mu), \infty)$. The next Lemma provides some invariant functions for the generator $\bar{L}$.

Lemma 7. *Let $\lambda, \mu \geq 0$. Define, for $x \in (q(\mu), \infty)$,*

$$g_{\lambda,\mu}(x) := \frac{1}{\Psi(x) - \mu} \exp\left[\int_\theta^x \frac{\Phi(u) + \lambda}{\Psi(u) - \mu} \mathrm{d}u\right], \tag{19}$$

and

$$f_{\lambda,\mu}(x) := \int_{q(\mu)}^\infty e^{-xz} g_{\lambda,\mu}(z) \mathrm{d}z.$$

If $\lambda > 0$, the function $f_{\lambda,\mu}$ is a C_0^2-function decreasing on (ℓ, ∞) such that

$$\bar{L} f_{\lambda,\mu} = \lambda f_{\lambda,\mu}.$$

Proof of Lemma 7. Let $\lambda > 0$, $\mu \geq 0$. Firstly, we check that $f_{\lambda,\mu}(x)$ is well-defined for $x > \ell$. We have

$$\frac{\Phi(u)}{\Psi(u) - \mu} = \frac{\Phi(u)}{u} \frac{u}{\Psi(u) - \mu} \underset{u\to\infty}{\longrightarrow} \frac{b}{\mathbf{d}} =: \ell,$$

therefore

$$\frac{1}{z} \int_\theta^z \frac{\Phi(u) + \lambda}{\Psi(u) - \mu} \mathrm{d}u \underset{z\to\infty}{\longrightarrow} \ell.$$

Since $x > \ell$ and $\Psi(z) - \mu \geq Cz$ with large enough z, and a constant $C > 0$, we get for all $\lambda \geq 0$

$$\int_\theta^\infty \frac{\mathrm{d}z}{\Psi(z)} \exp\left[-xz + \int_\theta^z \frac{\Phi(u) + \lambda}{\Psi(u) - \mu} \mathrm{d}u\right] < \infty. \tag{20}$$

It remains to verify the integrability at $q(\mu)$. We have

$$\int_{q(\mu)}^\theta \frac{\mathrm{d}z}{\Psi(z) - \mu} \exp\left[-xz - \int_z^\theta \frac{\Phi(u) + \lambda}{\Psi(u) - \mu} \mathrm{d}u\right] \leq \int_{q(\mu)}^\theta \frac{\mathrm{d}z}{\Psi(z) - \mu} \exp\left[-\int_z^\theta \frac{\lambda}{\Psi(u) - \mu} \mathrm{d}u\right].$$

Consider $\lambda > 0$, an antiderivative of the integrand on the right hand side is

$$z \mapsto \frac{1}{\lambda} \exp\left[-\lambda \int_z^\theta \frac{1}{\Psi(u) - \mu} \mathrm{d}u\right]. \tag{21}$$

This takes a finite value at $q(\mu)$ and yields the wished integrability. Similar arguments entails that $f_{\lambda,\mu}$ is twice-differentiable and tends to 0 when x goes to ∞.

Remark that $g_{\lambda,\mu}$ solves the ordinary differential equation

$$\forall z \in (q(\mu), \infty) \quad \Psi'(z) g_{\lambda,\mu}(z) + (\Psi(z) - \mu)\, g'_{\lambda,\mu}(z) = (\Phi(z) + \lambda) g_{\lambda,\mu}(z). \tag{22}$$

For all z, define $h_z(x) = e^{-xz}$, one can easily check that

$$\bar{L} h_z(x) = [x(\Psi(z) - \mu) - \Phi(z)]\, h_z(x).$$

We compute

$$\begin{aligned}\bar{L}f_{\lambda,\mu}(x) - \lambda f_{\lambda,\mu}(x) &= \int_{q(\mu)}^{\infty} \left(\bar{L}h_z(x) - \lambda h_z(x)\right) g_{\lambda,\mu}(z)dz \\ &= \int_{q(\mu)}^{\infty} e^{-xz}\left(x(\Psi(z)-\mu) - \Phi(z) - \lambda\right) g_{\lambda,\mu}(z)\mathrm{d}z \\ &= \int_{q(\mu)}^{\infty} e^{-xz}\left(\Psi'(z)g_{\lambda,\mu}(z) + (\Psi(z)-\mu)g'_{\lambda,\mu}(z) - (\Phi(z)+\lambda)g_{\lambda,\mu}(z)\right)\mathrm{d}z \\ &= 0.\end{aligned}$$

The third equality follows from integration by parts. Indeed, we have

$$(\Psi(x)-\mu)g_{\lambda,\mu}(x) = \exp\left(\int_{\theta}^{x} \frac{\Phi(u)+\lambda}{\Psi(u)-\mu}\mathrm{d}u\right) \underset{x\to q(\mu)}{\longrightarrow} 0$$

because $\int_{q(\mu)+} \frac{\mathrm{d}u}{\Psi(u)-\mu} = \infty$, since $\Psi(u)-\mu$ is always sub-linear near $q(\mu)$. The last equality holds true because of the ODE (22). □

We establish now Theorem 1.

Proof of Theorem 1. Consider a CBI(Ψ, Φ) process $(X_t, t \geq 0)$ and define $I_t := \int_0^t X_s \mathrm{d}s$. The family $(e^{-\mu I_t}, t \geq 0)$ is a continuous multiplicative functional of $(X_t, t \geq 0)$. Denote the subordinate semi-group (in the terminology of Blumenthal and Getoor [5]) by Q_t, and the subprocess by $(\bar{X}_t, t \geq 0)$. We have for all $f \in C^2(\mathbb{R}_+)$

$$Q_t f(x) = \bar{\mathbb{E}}[f(\bar{X}_t)] := \mathbb{E}_x[f(X_t)e^{-\mu I_t}].$$

We refer the reader to Theorems 3.3 and 3.12, pages 106 and 110 of Blumenthal and Getoor [5]. The bivariate process $((X_t, I_t); t \geq 0)$ is a Markov process. Similarly as Patie [25] (see Lemma 7), one can see by Itô's formula that for any function $f \in C_c^2(\mathbb{R}_+)$,

$$f(X_t)e^{-\mu\int_0^t X_s \mathrm{d}s} - f(x) - \int_0^t e^{-\mu\int_0^s X_u \mathrm{d}u}(Lf(X_s) - \mu X_s f(X_s))\mathrm{d}s$$

is a local martingale. Theorem 4.1.2 in [20] applies and ensures that $(\bar{X}_t, t \geq 0)$ is a CBI($\bar{\Psi}$, Φ) process. Firstly we consider $a > \ell$, and recall $\sigma_a = \inf\{t \geq 0, X_t = a\}$. From Lemma 7, one can apply Dynkin's formula to the Markov process $(\bar{X}_t, t \geq 0)$ killed at time σ_a, we get

$$\bar{\mathbb{E}}_x[e^{-\lambda\sigma_a\wedge t} f_{\lambda,\mu}(\bar{X}_{\sigma_a\wedge t})] = f_{\lambda,\mu}(x),$$

and thus

$$\mathbb{E}_x[e^{-\mu I_{\sigma_a\wedge t}} e^{-\lambda\sigma_a\wedge t} f_{\lambda,\mu}(X_{\sigma_a\wedge t})] = f_{\lambda,\mu}(x).$$

If we start from a point $x > a$, since the process has no downward jumps, $X_t > a$ for all time $t < \sigma_a$, and $f_{\lambda,\mu}(X_{t\wedge\sigma_a}) \leq f_{\lambda,\mu}(a)$. Therefore the left hand side of the above equality is bounded and when $t \to \infty$, we get

$$\mathbb{E}_x\left[\exp\left\{-\mu\int_0^{\sigma_a} X_t \mathrm{d}t - \lambda\sigma_a\right\}\right] = \frac{f_{\lambda,\mu}(x)}{f_{\lambda,\mu}(a)},$$

with the convention $e^{-\infty} = 0$. To prove the formula in the case $a = \ell$, we notice that σ_a is increasing towards σ_ℓ, when $a \downarrow \ell$, by quasi-left continuity of the CBI. The result follows by monotonicity. □

6. Hitting times and polarity of the boundary point

6.1. Hitting times

By a slight abuse of notation, define $f_\lambda := f_{\lambda,0}$ and $g_\lambda := g_{\lambda,0}$, that is to say

$$g_\lambda(x) = \frac{1}{\Psi(x)} \exp\left[\int_\theta^x \frac{\Phi(u)+\lambda}{\Psi(u)} \mathrm{d}u\right] \tag{23}$$

and $f_\lambda(x) = \int_{q(0)}^\infty e^{-xz} g_\lambda(z)\mathrm{d}z$. As a direct consequence of Theorem 1, when μ goes to 0, we get the following corollary.

Corollary 8. *For all $\lambda \in (0,\infty)$, and $x > a \geq \ell$*

$$\mathbb{E}_x\left[e^{-\lambda\sigma_a}\right] = \frac{f_\lambda(x)}{f_\lambda(a)}. \tag{24}$$

Remark 4. The process $(e^{-\lambda t} f_\lambda(X_t), t \geq 0)$ is not a martingale. This issue comes from the fact that f_λ is not in the domain of the generator associated to the CBI(Ψ, Φ) process. Indeed, we can plainly check that for any mechanisms Ψ, Φ: $|f_\lambda'(0)| = \infty$. In the same vein as scale functions for Lévy processes, one has to stop the process to get a martingale.

To the best of our knowledge these functions do not appear in the literature even when no immigration is taken into account. Consider that particular case and assume here that $\Phi \equiv 0$. The CBI is then a CB(Ψ) process. In the supercritical case, an easy calculation of the limit when λ goes to 0 yields

$$\mathbb{P}_x(\sigma_a < \infty) = \exp\left(-(x-a)q(0)\right), \quad \forall a \in]0,x].$$

Note that this equality holds for $a = 0$ under Grey's condition (3) (see for instance Theorem 3.8 in [21]). Furthermore, the function f_λ has a simpler expression. Indeed, since $\frac{z}{\Psi(z)} \to 1/\mathbf{d} \in [0,\infty)$ as $z \to \infty$, there exists $k > 0$ such that for $x > 0$,

$$e^{-xz} \exp\Big(\lambda \int_\theta^z \frac{\mathrm{d}u}{\Psi(u)}\Big) \leq e^{-xz} \exp\Big(\lambda k \int_\theta^z \frac{\mathrm{d}u}{u}\Big) = e^{-xz}(z/\theta)^{\lambda k} \underset{z\to\infty}{\longrightarrow} 0.$$

Since $\Psi'(q(0)) < \infty$, we also have $e^{-xz}\exp\Big(\lambda \int_\theta^z \frac{\mathrm{d}u}{\Psi(u)}\Big) \to 0$ as $z \to q(0)$. Performing integration by parts, a notable cancellation occurs, we get:

$$f_\lambda(x) = \frac{x}{\lambda}\int_{q(0)}^\infty e^{-xz} \exp\left(\lambda \int_\theta^z \frac{\mathrm{d}u}{\Psi(u)}\right) \mathrm{d}z,$$

and then for $x > a > \ell$,

$$\mathbb{E}_x[e^{-\lambda\sigma_a}] = \frac{x\int_{q(0)}^\infty e^{-xz}\exp\left(\lambda\int_\theta^z \frac{\mathrm{d}u}{\Psi(u)}\right)\mathrm{d}z}{a\int_{q(0)}^\infty e^{-az}\exp\left(\lambda\int_\theta^z \frac{\mathrm{d}u}{\Psi(u)}\right)\mathrm{d}z}.$$

We return to the general case for which $\Phi \not\equiv 0$. We end the section with a formula for the mean of the hitting time. When the branching mechanism is supercritical, we have $\mathbb{P}_x(\sigma_a = \infty) > 0$ for $x > a \geq \ell$. This follows readily by taking $\lambda \to 0$ in Corollary 8. Hence, we focus on the critical or subcritical case.

Corollary 9. *In the critical, or subcritical case, for all $x > a \geq \ell$,*

$$\mathbb{E}_x[\sigma_a] = \int_0^\infty \frac{\mathrm{d}z}{\Psi(z)} \left(e^{-az} - e^{-xz}\right) \exp\left(\int_0^z \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u\right). \tag{25}$$

Proof. Consider firstly $a > \ell$; the case $a = \ell$ will follow by monotonicity. One has

$$\mathbb{E}_x[\sigma_a] = \lim_{\lambda \to 0} \lambda^{-1} \left(1 - \mathbb{E}_x\left[e^{-\lambda \sigma_a}\right]\right) = \lim_{\lambda \to 0} \frac{f_\lambda(a) - f_\lambda(x)}{\lambda f_\lambda(a)}.$$

By integration by parts, we compute

$$\begin{aligned} \lambda f_\lambda(a) &= \int_0^\infty \mathrm{d}z \left(a - \frac{\Phi(z)}{\Psi(z)}\right) \exp\left(-az + \int_1^z \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u + \lambda \int_1^z \frac{\mathrm{d}u}{\Psi(u)}\right) \\ &\underset{\lambda \to 0}{\longrightarrow} \exp\left(-\int_0^1 \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u\right). \end{aligned} \tag{26}$$

Moreover, by monotonicity,

$$f_\lambda(a) - f_\lambda(x) \underset{\lambda \to 0}{\longrightarrow} \int_0^\infty \frac{\mathrm{d}z}{\Psi(z)} \left(e^{-az} - e^{-xz}\right) \exp\left(\int_1^z \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u\right). \tag{27}$$

The above integral is finite if and only if $\int_{0+} \frac{z}{\Psi(z)} \exp\left(\int_1^z \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u\right) \mathrm{d}z < \infty$. Since $z \leq C\Phi(z)$ at the neighborhood of 0, for some constant $C \in (0, \infty)$, we have plainly the integrability by anti-differentiation. When $\int_1^z \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u < \infty$ one can combine (26) and (27) to get the desired formula. When $\int_0^1 \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u = \infty$, we have $\mathbb{E}_x[\sigma_a] = \infty$. □

Remark 5. Note that for a CBI(Ψ, Φ) with $\Phi \not\equiv 0$ one has for all $a > 0$, $\mathbb{E}_x[\sigma_a] < \infty$ if and only if $\int_0^1 \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u < \infty$ (this is nothing but the positive recurrence of the process). The first moment of a CB(Ψ) process is finite if and only if $\int_0 \frac{z}{\Psi(z)} \mathrm{d}z < \infty$. One can plainly check those assertions for $a = 0$ in the non polar case.

6.2. Proof of Theorem 2: polarity of zero

When $\ell = 0$, Corollary 8 provides the Laplace transform of σ_0, the hitting time of 0. We study now the polarity of the boundary. Recall that a point $a \in \mathcal{S}$ is said to be polar if for all $x \in \mathcal{S}$ such as $x \neq a$,

$$\mathbb{P}_x(\sigma_a < \infty) = 0.$$

Let $\lambda > 0$. From Corollary 8, the point a is polar if and only if $f_\lambda(a) = \infty$. We have seen that $f_\lambda(x) \in (0, \infty)$ for any $x \in (\ell, \infty)$. Thus only ℓ may be polar. Firstly, if $\mathbf{d} < \infty$, note that

$$\frac{\Phi(x)}{\Psi(x)} - \ell = \frac{1}{b\Psi(x)} \left[\mathbf{d} \int_0^\infty (1 - e^{-xu}) \nu(\mathrm{d}u) + b \int_0^\infty (1 - e^{-xu}) \pi(\mathrm{d}u)\right] \geq 0.$$

Then

$$\int_\theta^\infty \frac{\mathrm{d}z}{\Psi(z)} \exp\left[-\ell z + \int_\theta^z \frac{\Phi(x)}{\Psi(x)} \mathrm{d}x\right] \geq e^{-\ell\theta} \int_\theta^\infty \frac{\mathrm{d}z}{\Psi(z)} = \infty,$$

and therefore we have $f_\lambda(\ell) = \infty$.

Assume now $\mathbf{d} = \infty$ (thus $\ell = 0$) and $\int_\theta^\infty \frac{\mathrm{d}z}{\Psi(z)} \exp\left[\int_\theta^z \frac{\Phi(x)}{\Psi(x)}\mathrm{d}x\right] = \infty$. We have $f_\lambda(0) = \infty$ and the same arguments hold.

We show now that if $\int_\theta^\infty \frac{\mathrm{d}z}{\Psi(z)} \exp\left[\int_\theta^z \frac{\Phi(x)}{\Psi(x)}\mathrm{d}x\right] < \infty$, then

$$\mathbb{P}_x[\sigma_0 < \infty] > 0. \tag{28}$$

Recall g_λ defined in (23). Using Corollary 8 for $a = 0$ and $\lambda \in (0, \infty)$, one gets

$$\mathbb{E}_x[e^{-\lambda\sigma_0}] = \frac{\int_{q(0)}^\theta e^{-xz} g_\lambda(z)\mathrm{d}z + \int_\theta^\infty e^{-xz} g_\lambda(z)\mathrm{d}z}{\int_{q(0)}^\theta g_\lambda(z)\mathrm{d}z + \int_\theta^\infty g_\lambda(z)\mathrm{d}z} \tag{29}$$

$$\geq \frac{e^{-x\theta} \int_{q(0)}^\theta g_\lambda(z)\mathrm{d}z}{\int_{q(0)}^\theta g_\lambda(z)\mathrm{d}z + \int_\theta^\infty g_\lambda(z)\mathrm{d}z} \tag{30}$$

$$\geq \frac{e^{-x\theta}}{1 + \int_\theta^\infty g_\lambda(z)\mathrm{d}z / \int_{q(0)}^\theta g_\lambda(z)\mathrm{d}z}. \tag{31}$$

The assumption entails that $\int_\theta^\infty g_\lambda(z)\mathrm{d}z \underset{\lambda\to 0}{\longrightarrow} \int_\theta^\infty g_0(z)\mathrm{d}z < \infty$. On the one hand, if $\int_{q(0)}^\theta g_0(z)\mathrm{d}z < \infty$, then, on the right hand-side of (31), the integrals tend to a positive and finite limit when λ goes to 0. On the other hand, if $\int_{q(0)}^\theta g_0(z)\mathrm{d}z = \infty$, the right-hand-side tends to $e^{-x\theta}$. In both cases, we deduce that $\mathbb{E}_x[e^{-\lambda\sigma_0}]$ decreases to a positive limit when λ decreases to 0, which entails (28).

7. Recurrence and transience

7.1. Infimum of a transient CBI and long-term behavior of a CB process conditioned to be non-extinct

Assume that the process $(X_t, t \geq 0)$ is transient. One can plainly check that the function

$$f_0(x) = \int_{q(0)}^\infty \frac{\mathrm{d}z}{\Psi(z)} \exp\left(-xz + \int_\theta^z \frac{\Phi(u)}{\Psi(u)}\mathrm{d}u\right)$$

takes finite values for all $x > \ell$. Applying Corollary 8 and Theorem 3, we obtain the following proposition.

Proposition 10. *Denote the overall infimum of the transient process $(X_t, t \geq 0)$ by I. We have*

$$\mathbb{P}_x(I \leq a) = \mathbb{P}_x(\sigma_a < \infty) = \frac{f_0(x)}{f_0(a)}.$$

If $f_0(0) = \int_{q(0)}^\infty \frac{\mathrm{d}z}{\Psi(z)} \exp\left(\int_\theta^z \frac{\Phi(u)}{\Psi(u)}\mathrm{d}u\right) < \infty$ (i.e 0 is not polar and the process is transient) then the law of I has an atom at 0.

Proof. Firstly, note that $\mathbb{P}_x[I \leq a] = \mathbb{P}_x[\sigma_a < \infty]$. By Theorem 3, the integrability condition needed to define f_0 is satisfied. Taking $\lambda = 0$, in the formula for the Laplace transform of σ_a, yields $\mathbb{P}_x[\sigma_a < \infty] = \frac{f_0(x)}{f_0(a)}$. □

The CB(Ψ) process conditioned to be non extinct corresponds to a CBI(Ψ, Φ) process with $\Phi(q) = \Psi'(q) - \Psi'(0+)$. As a direct corollary of Theorem 3, we recover and complete some results due to Lambert (see Theorem 4.2(i) in [18]). In the (sub)critical case, $q(0) = 0$ and we choose $\theta = 1$.

Corollary 11. *The critical CB process conditioned to be non extinct is transient. Moreover, if the process starts at x, its minimum is uniformly distributed over* $[0, x]$. *The subcritical* CB(Ψ) *process conditioned to be non extinct is recurrent or transient according to*

$$\int_0^1 \frac{\mathrm{d}z}{z} \exp\left(-\int_z^1 \left(\frac{1}{\Psi'(0+)u} - \frac{1}{\Psi(u)}\right) \mathrm{d}u\right) = \infty \text{ or } < \infty.$$

Proof. Let Ψ be a critical reproduction mechanism. Consider the case $\Phi = \Psi'$, the CBI(Ψ, Φ) process has the same law as the CB(Ψ) process conditioned to the non extinction. In that case, we have clearly $\int_1^z \frac{\Psi'(u)}{\Psi(u)} \mathrm{d}u = \log(\Psi(z)) - \log(\Psi(1))$, and therefore

$$\int_0^1 \frac{\mathrm{d}z}{\Psi(z)} \exp\left(\int_1^z \frac{\Psi'(u)}{\Psi(u)} \mathrm{d}u\right) = \frac{1}{\Psi(1)} < \infty.$$

In order to deal with the minimum, one can readily check that $f_0(x) = 1/x$. Thus, the random variable I is uniformly distributed over $[0, x]$. For the subcritical case, plugging $\Phi = \Psi' - \Psi'(0+)$ in the integral of Theorem 3, yields easily the statement. □

Remark 6. The fact that the minimum of a critical CBI(Ψ, Ψ') is uniformly distributed can be obtained alternatively from Proposition 3 in Chaumont [7], which states the corresponding result for Lévy processes conditioned to stay positive. Indeed, Lambert, in [17], shows that the CB process conditioned to be non-extinct has the same law as a time-changed Lévy process conditioned to stay positive.

7.2. *Proofs of Theorem 3 and Corollary 4*

7.2.1. *Proof of Theorem 3*

Firstly, we establish statement (a) of Theorem 3. The proof relies on the study of the Laplace transform of the hitting times provided by Corollary 8. Recall

$$g_\lambda(x) = \frac{1}{\Psi(x)} \exp\left[\int_1^x \frac{\Phi(u) + \lambda}{\Psi(u)} \mathrm{d}u\right].$$

Recurrence. Assume that

$$\int_0^1 \frac{1}{\Psi(x)} \exp\left[-\int_x^1 \frac{\Phi(u)}{\Psi(u)} \mathrm{d}u\right] \mathrm{d}x = \infty. \tag{32}$$

For every $x \geq a$,

$$\mathbb{P}_x[\sigma_a < \infty] = \lim_{\lambda \to 0} \mathbb{E}_x[e^{-\lambda\sigma_a}].$$

By Corollary 8, we have for all $a > \ell$, for all $\varepsilon \in (0, \infty)$,

$$\mathbb{E}_x[e^{-\lambda\sigma_a}] = \frac{\int_0^\varepsilon e^{-xz} g_\lambda(z)\mathrm{d}z + \int_\varepsilon^\infty e^{-xz} g_\lambda(z)\mathrm{d}z}{\int_0^\varepsilon e^{-az} g_\lambda(z)\mathrm{d}z + \int_\varepsilon^\infty e^{-az} g_\lambda(z)\mathrm{d}z} \tag{33}$$

$$\geq \frac{e^{-x\varepsilon}\int_0^\varepsilon g_\lambda(z)\mathrm{d}z}{\int_0^\varepsilon g_\lambda(z)\mathrm{d}z + \int_\varepsilon^\infty e^{-az} g_\lambda(z)\mathrm{d}z} \tag{34}$$

$$\geq \frac{e^{-x\varepsilon}}{1 + \int_\varepsilon^\infty e^{-az} g_\lambda(z)\mathrm{d}z / \int_0^\varepsilon g_\lambda(z)\mathrm{d}z}. \tag{35}$$

The assumption (32) entails that $\int_0^\varepsilon g_\lambda(z)\mathrm{d}z \to \infty$, as λ goes to 0. On the other hand, Eq. (20) ensures that $\int_\varepsilon^\infty e^{-az} g_0(z)\mathrm{d}z < \infty$, because $a > \ell$. Therefore, the right-hand-side in (35) tends to $e^{-x\varepsilon}$, as $\lambda \to 0$. The real number ε can be chosen arbitrarily small, so we get

$$\mathbb{E}_x[e^{-\lambda\sigma_a}] \underset{\lambda\to 0}{\longrightarrow} 1.$$

We deduce that $\mathbb{P}_x(\sigma_a < \infty) = 1$ for any $x \geq a > \ell$, which implies $\mathbb{P}_x\left(\liminf_{t\to\infty} X_t \leq \ell\right) = 1$. The lower bound provided in Proposition 5 then entails $\mathbb{P}_x\left(\liminf_{t\to\infty} X_t = \ell\right) = 1$, so the process is recurrent in the sense of (1).

***Transience*.** We now work under the assumption

$$\int_0^1 \frac{\mathrm{d}z}{\Psi(z)} \exp\left(-\int_z^1 \frac{\Phi(u)}{\Psi(u)}\mathrm{d}u\right) < \infty. \tag{36}$$

Let $a > \ell$. We show that $\mathbb{P}_x\left(\liminf_{t\to\infty} X_t < a\right) = 0$. One has

$$\mathbb{P}_x\left(\liminf_{t\to\infty} X_t < a\right) \leq \lim_{t\to\infty} \mathbb{P}_x\left(\sigma_a \circ \theta_t < \infty\right)$$

$$= \lim_{t\to\infty} \mathbb{E}_x\left[\mathbb{P}_{X_t}\left(\sigma_a < \infty\right)\right]. \tag{37}$$

Moreover one can write that,

$$\mathbb{E}_x\left[\mathbb{P}_{X_t}\left(\sigma_a < \infty\right)\right] \leq \mathbb{P}_x\left(X_t \leq a\right) + \mathbb{E}_x\left[\mathbf{1}_{\{X_t > a\}}\mathbb{P}_{X_t}\left(\sigma_a < \infty\right)\right]. \tag{38}$$

Firstly, under (36), one has $\int_0^1 \frac{\Phi(u)}{\Psi(u)}\mathrm{d}u = \infty$. According to (ii) in Theorem 6, it implies that

$$\lim_{t\to 0} \mathbb{P}_x\left(X_t \leq a\right) = 0.$$

Thus, the first term in (38) goes to 0 when $t \to \infty$. We focus now on the second term. Under (36), one can take $\lambda = 0$ in Corollary 8. For $x > a > \ell$,

$$\mathbb{P}_x\left(\sigma_a < \infty\right) = \frac{\int_0^\infty \frac{\mathrm{d}z}{\Psi(z)} \exp\left(-xz + \int_1^z \frac{\Phi(u)}{\Psi(u)}\mathrm{d}u\right)}{\int_0^\infty \frac{\mathrm{d}z}{\Psi(z)} \exp\left(-az + \int_1^z \frac{\Phi(u)}{\Psi(u)}\mathrm{d}u\right)} = c_a \int_0^\infty g_0(z) e^{-xz}\mathrm{d}z \tag{39}$$

with $c_a = \left(\int_0^\infty g_0(z)e^{-az}\mathrm{d}z\right)^{-1}$. Hence,

$$\begin{aligned}\mathbb{E}_x\left[\mathbf{1}_{\{X_t>a\}}\mathbb{P}_{X_t}(\sigma_a<\infty)\right] &= c_a\mathbb{E}_x\left[\mathbf{1}_{\{X_t>a\}}\int_0^\infty g_0(z)e^{-zX_t}\mathrm{d}z\right]\\ &= c_a\int_0^\infty g_0(z)\mathbb{E}_x\left[\mathbf{1}_{\{X_t>a\}}e^{-zX_t}\right]\mathrm{d}z. \qquad (40)\end{aligned}$$

Moreover, $\mathbb{E}_x\left[\mathbf{1}_{\{X_t>a\}}e^{-zX_t}\right]\le e^{-za}$ and by (36) and (20), $\int_0^\infty g_0(z)e^{-za}\mathrm{d}z<\infty$. Furthermore, since $\int_{0+}\frac{\Phi(u)}{\Psi(u)}\mathrm{d}u=\infty$,

$$\mathbb{E}_x\left[\mathbf{1}_{\{X_t>a\}}e^{-zX_t}\right]\le\mathbb{E}_x\left[e^{-zX_t}\right]=\exp\left(-xv_t(z)-\int_{v_t(z)}^z\frac{\Phi(u)}{\Psi(u)}\mathrm{d}u\right)\underset{t\to\infty}{\longrightarrow}0. \qquad (41)$$

Thus, by dominated convergence, the integral (40) tends to 0, which entails the desired result. Therefore the process is transient in the sense of Definition (2).

In order to prove statement (b) (transience in the supercritical case), one has just to adapt the proof above. Indeed, we have $\int_{q(0)}^\theta \frac{\mathrm{d}z}{\Psi(z)}\exp\left(-\int_z^\theta\frac{\Phi(u)}{\Psi(u)}\mathrm{d}u\right)<\infty$, so we can write (39). Moreover, one can use that $\int_{v_t(z)}^z\frac{\Phi(u)}{\Psi(u)}\mathrm{d}u\underset{t\to\infty}{\longrightarrow}\int_{q(0)}^z\frac{\Phi(u)}{\Psi(u)}\mathrm{d}u=\infty$ in (41). □

7.2.2. Proof of Corollary 4

Recall Corollary 4. We assume $\int_1^\infty u^2\nu(\mathrm{d}u)<\infty$, we have then

$$\begin{aligned}\Phi(q) &= \left(b+\int_0^\infty u\nu(\mathrm{d}u)\right)q+\int_0^\infty(1-e^{-qu}-qu)\nu(\mathrm{d}u)\\ &= \tilde{b}q+O(q^2)\quad\text{as } q\to0.\end{aligned}$$

On the other hand

$$\Psi(q)=\frac{1}{2}\tilde{\sigma}^2q^2-\int_0^\infty\left(1-e^{-qu}-qu+\frac{1}{2}q^2u^2\right)\pi(\mathrm{d}u). \qquad (42)$$

Set

$$h(q):=\int_0^\infty\left(1-e^{-qu}-qu+\frac{1}{2}q^2u^2\right)\pi(\mathrm{d}u).$$

The function h is non-negative and increasing. Consider

$$\begin{aligned}h_1(q) &= \int_0^1\left(1-e^{-qu}-qu+\frac{1}{2}q^2u^2\right)\pi(\mathrm{d}u)\\ h_2(q) &= \int_1^\infty\left(1-e^{-qu}-qu+\frac{1}{2}q^2u^2\right)\pi(\mathrm{d}u).\end{aligned}$$

It is easy to see that $h_1(q)=O(q^3)$ as $q\to0$. Moreover, by assumption $\int_1^\infty u^2\log u\,\pi(\mathrm{d}u)<\infty$ and it follows from Bingham et al. [4, Theorem 8.1.8, page 335] (with $n=1$ and $\beta=2$) that $\int_0^1 h_2(q)q^{-3}\mathrm{d}q<\infty$. Then we have

$$\int_0^1 h(q)q^{-3}\mathrm{d}q<\infty, \qquad (43)$$

which implies that $h(q) = o(q^2)$ as $q \to 0$. The following limit is finite and positive:

$$\lim_{q\to 0} \frac{\frac{1}{\Psi(q)} \exp\left[-\int_q^1 \frac{\Phi(x)}{\Psi(x)}\mathrm{d}x\right]}{\frac{1}{q^2} \exp\left[\int_q^1 \frac{\tilde{b}x}{\frac{1}{2}\tilde{\sigma}^2 x^2}\mathrm{d}x\right]} = \frac{2}{\tilde{\sigma}^2} \exp\Big[-\int_0^1 \Big(\frac{\Phi(x)}{\Psi(x)} - \frac{2\tilde{b}x}{\tilde{\sigma}^2 x^2}\Big)dx\Big] \in]0, \infty[. \tag{44}$$

Indeed a simple calculation yields

$$\begin{aligned}\int_0^1 \Big|\frac{\Phi(x)}{\Psi(x)} - \frac{2\tilde{b}x}{\tilde{\sigma}^2 x^2}\Big|\mathrm{d}x &\le \int_0^1 \Big|\frac{\Phi(x) - \tilde{b}x}{\Psi(x)}\Big|\mathrm{d}x + \frac{2\tilde{b}}{\tilde{\sigma}^2}\int_0^1 \frac{h(x)}{\Psi(x)x}\mathrm{d}x \\ &= J_1 + J_2,\end{aligned}$$

and since $|\Phi(q) - \tilde{b}q| = O(\Psi(q))$ as $q \to 0$, we have $J_1 < \infty$, and (43) ensures that $J_2 < \infty$. Thus, it follows from (44) that

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}z}{\Psi(z)} \exp\left[-\int_z^1 \frac{\Phi(x)}{\Psi(x)}\mathrm{d}x\right] < \infty &\iff \int_0^1 \frac{1}{z^2} \exp\left[-\int_z^1 \frac{2\tilde{b}}{\tilde{\sigma}^2 x}\mathrm{d}x\right] < \infty \\ &\iff 2\tilde{b} > \tilde{\sigma}^2.\end{aligned}$$

7.3. *Construction of subcritical null-recurrent CBI processes*

We look here for examples of null recurrent CBI processes. Assume that $\Psi(q) = \gamma q$, with $\gamma > 0$. The following computations remain valid for any general subcritical mechanism Ψ, because only the behavior of Ψ at 0 matters. To avoid positive recurrence, we need to choose Φ such that $\int_0 \frac{\Phi(q)}{q}\mathrm{d}q = \infty$, which is equivalent to

$$\int^\infty \log(u)\nu(\mathrm{d}u) = \infty. \tag{45}$$

Moreover, to get a recurrent process, we know from Theorem 3 that Φ has to satisfy

$$\int_0^1 \frac{\mathrm{d}z}{z} \exp\left(-\int_z^1 \frac{\Phi(u)}{\gamma u}\mathrm{d}u\right) = \infty. \tag{46}$$

From condition (45), we know that the example of Φ we are looking for is not a deterministic drift. Moreover, when ν is not null, the value of the drift coefficient b has no influence for (46) to be fulfilled. Therefore, we will take $b = 0$ and we will exhibit a sufficient condition involving the Lévy measure ν to get (46). Denote $\overline{\nu}(u) := \nu([u, \infty))$ and recall from Chapter III of Bertoin [1] that there exists a universal constant κ such that

$$\Phi(q)/q \le \kappa J_\Phi(1/q), \quad \forall q > 0, \quad \text{where } J_\Phi(x) := \int_0^x \overline{\nu}(u)\mathrm{d}u, \quad x > 0.$$

Thus, we have

$$\begin{aligned}\int_z^1 \frac{\Phi(u)}{\gamma u}\mathrm{d}u \le \frac{\kappa}{\gamma}\int_z^1 J_\Phi(1/u)\mathrm{d}u &= \frac{\kappa}{\gamma}\int_1^{1/z} J_\Phi(u)/u^2\mathrm{d}u \\ &= \frac{\kappa}{\gamma}\left(J_\Phi(1) - zJ_\Phi(1/z) + \int_1^{1/z} \overline{\nu}(u)/u\mathrm{d}u\right),\end{aligned}$$

by integration by parts. Hence, a sufficient condition to get (46) is

$$\int_0^1 \frac{\mathrm{d}z}{z} \exp\left(-\frac{\kappa}{\gamma}\int_1^{1/z} \overline{\nu}(u)/u\mathrm{d}u\right) = \infty. \tag{47}$$

Example 2. We consider $\alpha \in \mathbb{R}$ and define ν such that

$$\int_1^{1/z} \overline{\nu}(u)/u\mathrm{d}u = \alpha \log\log 1/z \text{ up to an add. constant.}, \tag{48}$$

so that the integral in (47) is of the same nature as $\int_0 \frac{\mathrm{d}z}{z\log(1/z)^{\kappa\alpha/\gamma}}$. The integral will be infinite if α is chosen such that $\kappa\alpha/\gamma \le 1$. We can get (48) taking $\overline{\nu}(u) := \alpha u \frac{\mathrm{d}}{\mathrm{d}u}\log\log u = \frac{\alpha}{\log u}$ on $[100, \infty]$, that is $\nu(\mathrm{d}u) = \frac{\alpha}{u\log^2 u}\mathbf{1}_{[100,\infty]}\mathrm{d}u$. We can easily check that ν is a Lévy measure and that the condition (45) is satisfied. This example is related to that given by Sato and Yamazato in Section 7 of [28], in which the authors observe that the null recurrence or transience of this process is a function of κ/γ. The form of the criterion (46) and the role played by Bertrand's integrals provide a better understanding of the criterion. In the next example, the value of γ has no influence.

Example 3. We choose ν such that

$$\int_1^{1/z} \overline{\nu}(u)/u\mathrm{d}u = \log\log\log 1/z \text{ up to an add. constant.}, \tag{49}$$

so that the integral in (46) is $\int_0 \frac{\mathrm{d}z}{z\log\log(1/z)^{\kappa/\gamma}} = \infty$. We can get (49) taking

$$\overline{\nu}(u) := u\frac{\mathrm{d}}{\mathrm{d}u}\log\log\log u = \frac{1}{\log u \log\log u}, \quad \text{on } [100, \infty[,$$

that is $\nu(\mathrm{d}u) = \frac{\log\log(u)+1}{u\log^2 u \log^2(\log u)}\mathbf{1}_{[100,\infty]}\mathrm{d}u$. The density of the last Lévy measure is equivalent at ∞ to $\frac{1}{u\log^2 u \log\log u}$. Hence, we can check that ν is indeed a Lévy measure and that it satisfies (45).

8. Total population

As already said, one can see the integral $\int_0^{\sigma_a} X_s\mathrm{d}s$ as the total population up to time σ_a. In the case of the CB(Ψ) ($\Phi \equiv 0$), and $a = 0$, this is known as the total progeny. Lamperti [19] observed that a CB(Ψ) process is connected to a spectrally positive Lévy process with Laplace exponent Ψ by a simple time-change $\int_0^t X_s\mathrm{d}s$. This allows one to transfer the study of $\int_0^{\sigma_a} X_s\mathrm{d}s$ to that of the hitting time of a Lévy process. See Bingham [3] and Corollary 10.9 in Kyprianou [16]. In what follows, we recover the latter corollary and obtain its analogue with immigration.

Proposition 12. *Let $x > a \ge \ell$, and assume that $\Phi \equiv 0$. For all $\mu > 0$,*

$$\mathbb{E}_x\left[\exp\left\{-\mu\int_0^{\sigma_a} X_t\mathrm{d}t\right\}\right] = \exp\left(-(x-a)q(\mu)\right). \tag{50}$$

Proof. Firstly, let $\lambda > 0$. By integration by parts, we have

$$\int_{q(\mu)}^{\infty} \frac{\mathrm{d}z}{\Psi(z)-\mu} \exp\left(-xz + \int_{\theta}^{z} \frac{\lambda}{\Psi(u)-\mu}\mathrm{d}u\right)$$
$$= \int_{q(\mu)}^{\infty} \mathrm{d}z\, xe^{-xz} \exp\left(-xz + \int_{\theta}^{z} \frac{\lambda}{\Psi(u)-\mu}\mathrm{d}u\right),$$

which tends to $\int_{q(\mu)}^{\infty} \mathrm{d}z\, xe^{-xz} = \exp(-xq(\mu))$, as λ goes to 0. Thus, let $a > \ell \geq 0$. The desired result follows from Theorem 1, with $\Phi \equiv 0$, letting $\lambda \to 0$. One can obtain now the case $a = \ell$ by monotonicity and quasi-left continuity. □

More generally, we have the following corollary of Theorem 1.

Corollary 13. *Let $x > a \geq \ell$, and assume that $\Phi \not\equiv 0$. For all $\mu > 0$,*

$$\mathbb{E}_x\left[\exp\left\{-\mu \int_0^{\sigma_a} X_t \mathrm{d}t\right\}\right] = \frac{\int_{q(\mu)}^{\infty} \frac{\mathrm{d}z}{\Psi(z)-\mu} \exp\left(-xz + \int_{\theta}^{z} \frac{\Phi(u)}{\Psi(u)-\mu}\mathrm{d}u\right)}{\int_{q(\mu)}^{\infty} \frac{\mathrm{d}z}{\Psi(z)-\mu} \exp\left(-az + \int_{\theta}^{z} \frac{\Phi(u)}{\Psi(u)-\mu}\mathrm{d}u\right)}. \tag{51}$$

In the particular case of the CBI(Ψ, Ψ') *with* $\Psi'(0+) = 0$ *(this is the* CB(Ψ) *conditioned to be non extinct), we have*

$$\mathbb{E}_x\left[\exp\left\{-\mu \int_0^{\sigma_a} X_t \mathrm{d}t\right\}\right] = \frac{a}{x} \exp\left(-(x-a)q(\mu)\right), \quad \forall \mu > 0, x > a \geq \ell.$$

Proof. It follows readily from Theorem 1 by letting $\lambda \to 0$. We only have to check that the integral in the numerator of (51) is finite. At infinity, this follows from (20). At $q(\mu)$, one can use that

$$\Psi'(z) - \mu \underset{z\to q(\mu)}{\sim} \Psi'(q(\mu))\,(z - q(\mu)), \quad \text{and}$$
$$\int_{\theta}^{z} \frac{\Phi(u)}{\Psi(u)-\mu} du \underset{z\to q(\mu)}{\sim} \frac{\Phi(q(\mu))}{\Psi'(q(\mu))} \log\left(\frac{z-q(\mu)}{\theta-q(\mu)}\right),$$

where $\Phi(q(\mu))$ and $\Psi'(q(\mu)) \in (0, \infty)$ because $\mu \in (0, \infty)$. □

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